TENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 11, 2006

9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

1. GP identity.

Show that if the nonzero numbers a, b, c are in geometric progression, and $a^{-3}+b^{-3}+c^{-3} \neq 0$, then

$$\frac{a^3+b^3+c^3}{a^{-3}+b^{-3}+c^{-3}} = a^2b^2c^2.$$

2. An integral.

Evaluate

$$\int_1^{2006} \frac{dx}{x + \lfloor \log_{10} x \rfloor}.$$

(For a real number u, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.)

3. A simple product.

Express the product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2006^2}\right)$$

as simply as you can as a rational fraction in lowest terms; i.e., in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Justify your answer.

4. Sums of powers of 3.

How many of the integers from 1 to 2006 may be written as the sum of two or more distinct integral powers of 3? (For example, $28 = 3^0 + 3^3$ is such an integer.) Justify your answer.

5. An integer combination.

Do there exist integers a, b, c, with $abc \neq 0$, such that $2006a + 206b + 26c = 0$? If so, find one such triple (a, b, c); if none exist, prove it.

6. A cube and its reciprocal.

If r is a real number and $\sqrt[3]{r} + \dfrac{1}{\sqrt[3]{r}} = 3$, find, with proof, the value of $r^3 + \dfrac{1}{r^3}$.

7. The largest term.

When $(1 + 0.1)^{2006}$ is expanded by the binomial theorem, a sum of 2007 terms is obtained:

$$\binom{2006}{0}(0.1)^0+\binom{2006}{1}(0.1)^1+\binom{2006}{2}(0.1)^2+\cdots+\binom{2006}{2006}(0.1)^{2006} = T_0+T_1+T_2+\cdots+T_{2006},$$

where $T_k = \binom{2006}{k}(0.1)^k$. Which of the terms $T_k,\ k = 0, 1, 2, \ldots, 2006$, is the largest?

8. Integral roots.

Find all real numbers b such that both roots of the equation $x^2 + bx + 5b = 0$ are integers. Defend your answer.

9. A special rational approximation.

Prove that there exist positive integers m and n such that

$$\left|\frac{m^2}{n^3} - \sqrt{2006}\right| < 10^{-8}.$$

10. Infinitely many square terms.

Prove that the arithmetic progression $\{19 + 2006n : n \in \boldsymbol{N}\} = \{19, 2025, 4031, \cdots\}$ has infinitely many terms which are squares of integers.