Putnam Exam Problems

November 20, 1965, A-6: In the plane with orthogonal Cartesian coordinates $x$ and $y$, prove that the line whose equation is $ux + vy = 1$ will be tangent to the curve $x^m + y^m = 1$ (where $m > 1$) if and only if $u^n + v^n = 1$ and $m^{-1} + n^{-1} = 1$.

November 19, 1966, A-1: Let $f(n)$ be the sum of the first $n$ terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots$$

Show that if $x > y > 0$ then $xy = f(x + y) - f(x - y)$.

November 19, 1966, A-6: Justify the statement that

$$3 = \sqrt{1 + 2} \sqrt{1 + 3} \sqrt{1 + 4} \sqrt{1 + 5} \sqrt{1 + \cdots}$$

December 6, 1969, A-2: Let $D_n$ be the determinant of order $n$ of which the element in the $i^{th}$ row and the $j^{th}$ column is $|i - j|$. Prove

$$D_n = (-1)^{n-1}(n-1)2^{n-2}.$$ 

December 6, 1969, A-4: Prove

$$\int_0^1 x^n \, dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}$$

December 2, 1972, A-4: Of all ellipses inscribed in a square, show that the circle has the maximum perimeter.

December 2, 1972, A-5: Show that if $n > 1$ is an integer then $n$ does not divide $2^n - 1$.

December 3, 1977, A-1: Consider all lines which meet the graph of $y = 2x^4 + 7x^3 + 3x - 5$ in four distinct points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. Show that the average of the $x$ coordinates is independent of the line and find its value.

December 5, 1981, A-1: Let $E(n)$ denote the largest integer $k$ such that $5^k$ divides $1^12^23^3 \cdots n^n$. Calculate $\lim_{n \to \infty} \frac{E(n)}{n^n}$. 