Putnam Exam Problems

November 20, 1965, A-6: In the plane with orthogonal Cartesian coordinates x and y, prove that the line whose equation is $ux+vy=1$ will be tangent to the curve $x^m+y^m=1$ (where $m>1$) if and only if $u^n+v^n=1$ and $m^{-1}+n^{-1}=1$.

November 19, 1966, A-1: Let $f(n)$ be the sum of the first n terms of the squence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots$$

Show that if $x>y>0$ then $xy=f(x+y)-f(x-y)$.

November 19, 1966, A-6: Justify the statement that

$$3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}$$

December 6, 1969, A-2: Let D_n be the determinant of order n of which the element in the i^{th} row and the j^{th} column is $|i-j|$. Prove

$$D_n=(-1)^{n-1}(n-1)2^{n-2}.$$

December 6, 1969, A-4: Prove

$$\int_0^1 x^x\,dx=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-n}$$

December 2, 1972, A-4: Of all ellipses inscribed in a square, show that the circle has the maximum perimeter.

December 2, 1972, A-5: Show that if $n>1$ is an integer then n does not divide 2^n-1.

December 3, 1977, A-1: Consider all lines which meet the graph of $y=2x^4+7x^3+3x-5$ in four distinct points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4). Show that the average of the x coordinates is independent of the line and find its value.

December 5, 1981, A-1: Let $E(n)$ denote the largest integer k such that 5^k divides $1^1 2^2 3^3 \cdots n^n$. Calculate $\lim_{n\to\infty}\frac{E(n)}{n^2}$.