The 48th Annual High School Mathematics Contest
April 13, 2022
Minnesotata State University, Mankato
In memory of Dr. Waters who loved math and the math contest.

Instruction for the Contest

1. Write your answer on the **answer sheet**.

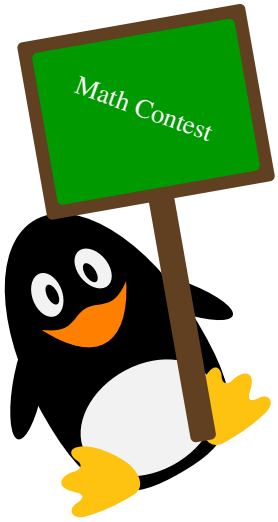
2. You have 90 minutes to work on 30 questions and one tie breaker problem.

3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.

4. You need to write a solution for the tie breaker problem on the **answer sheet**, and the tie breaker will be used only to break any possible ties that arise on the test.

5. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

6. **Submit your answer sheet** at the end of the contest and keep the exam booklet.



1. Two integer numbers between 1 and 9, inclusive, are selected at random. The same number may be selected twice. What is the probability that their product is a multiple of 3?

a) 36/81

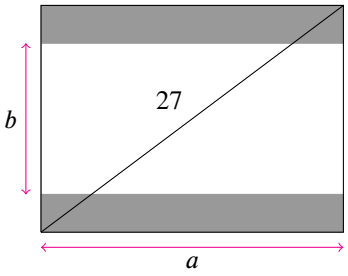
b) 5/9

c) 5/8

d) 25/81

e) 20/81

2. Older television screens have an aspect ratio of 4 : 3. That is, the ratio of the width to the height is 4 : 3. The aspect ratio of many movies is not 4 : 3, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown below.



$a : b = 2 : 1$

Suppose a movie has an aspect ratio of 2 : 1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?

a) 2

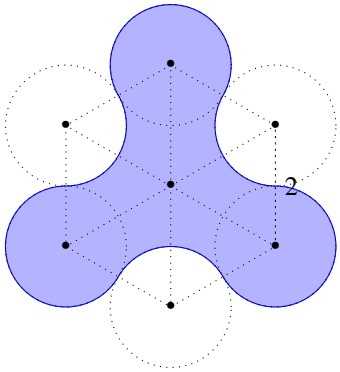
b) 2.7

c) 2.5

d) 2.25

e) 3

3. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is on a vertex of a regular hexagon of side length 2.



What is the area enclosed by the curve?

a) $2\pi + 6$

b) $2\pi + 4\sqrt{3}$

c) $3\pi + 4$

d) $2\pi + 2\sqrt{3} + 2$

e) $\pi + 6\sqrt{3}$

4. Alice has taken 4 tests in her mathematics class. If her score on the next test is 100, then her average after taking all five tests will be 90. What is her average score *right now* (only considering the first four tests)?

a) 84.8

b) 85

c) 86.25

d) 87.5

e) 89.75

5. How many positive integers n are there so that $1! + 2! + 3! + \cdots + n! \leq n^3$?

(The factorial $n!$ means the product of the first n positive integers, that is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2) \cdot (n-1) \cdot n. \qquad)$$

a) 1

b) 2

c) 3

d) 4

e) 5

6. Suppose that $a, b, 20, c, d$ is an arithmetic sequence. What is the value of $a + b + c + d$?

a.) 40

b.) 60

c.) 70

d.) 80

e.) 90

7. Bob goes onto a game show where the contestant gains 10 dollars for every right answer, but loses 6 dollars for every wrong answer. Bob answered 32 questions and left with 0 dollars. How many questions did he answer correctly?

a) 6

b) 10

c) 12

d) 13

e) 15

8. Find the sum

$$\frac{1}{\sqrt{9}+\sqrt{8}}+\frac{1}{\sqrt{8}+\sqrt{7}}+\frac{1}{\sqrt{7}+\sqrt{6}}+\frac{1}{\sqrt{6}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{4}}.$$

a) 2

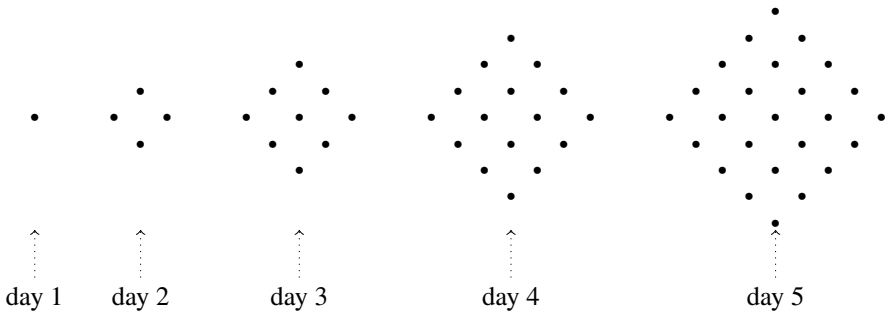
b) 3

c) 5

d) 1

e) $\sqrt{5}-\sqrt{2}$

9. On the first day Lisa  draws 1 dot, on day 2 she draws 4 dots, on day 3 she draws 9 dots and she continues drawing dots in this way.



On day 17, how many dots does she draw?

a) 144

b) 215

c) 289

d) 342

e) 437

10. If $f(x) = \frac{2x+3}{5x-7}$ then the inverse function $f^{-1}(x)$ of $f(x)$ is

a) $f^{-1}(x) = \frac{3x+5}{7x-2}$

b) $f^{-1}(x) = \frac{2x-7}{5x+3}$

c) $f^{-1}(x) = \frac{5x+2}{3x-7}$

d) $f^{-1}(x) = \frac{7x+3}{5x-2}$

e) $f^{-1}(x) = \frac{5x-7}{2x+3}$

11. Compute the number of ways in which 20 one-dollar bills can be distributed to 4 people so that no person receives less than 4.

a) 35

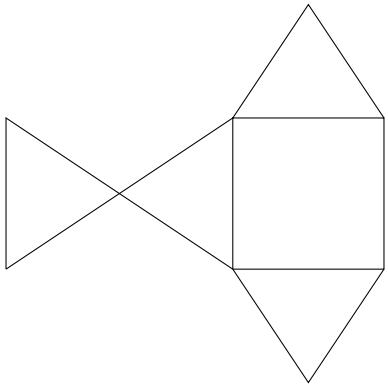
b) 24

c) 15

d) 30

e) 10

12. Suppose that the following figure is folded up into a three dimensional object



What sort of solid will it make?

a) Tetrahedron (Triangular pyramid)

b) Rectangular prism (Box)

c) Triangular prism

d) Square pyramid

e) Cone

13. The equation $x^2 + 2x + y^2 + 6y + 6 = 0$ defines a circle. What is its radius?

a) 2

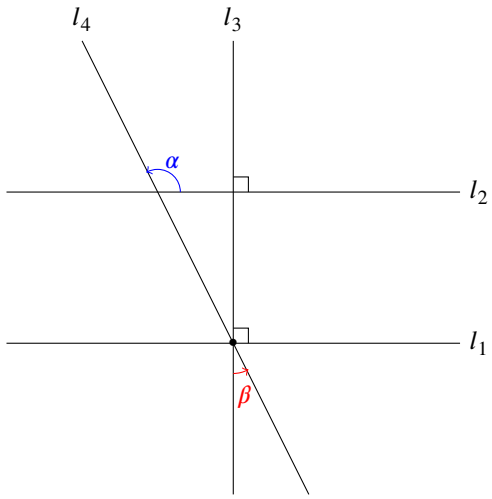
b) $\sqrt{6}$

c) $\sqrt{14}$

d) 4

e) None of the above

14. Consider the following picture



The line l_1 is parallel to the line l_2, and the line l_3 is perpendicular to the line l_1. The angle $\angle\alpha$ measures 110° (the picture is not to scale). What is the measure of $\angle\beta$?

a) 20°

b) 30°

c) 35°

d) 40°

e) 45°

15. A teacher with a math class of 20 students randomly pairs the students to take a test. What is the probability that Camilla and Cameron, two students in the class, are paired with each other?

a) 3/25

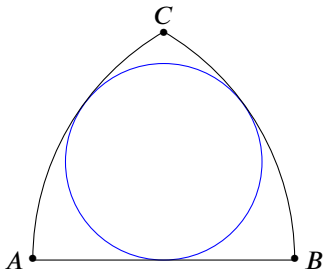
b) 1/24

c) 1/18

d) 1/20

e) 1/19

16. If circular arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ have centers at B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$.



If the length of $\overset{\frown}{BC}$ is 12, then the circumference of the circle is

a) 24

b) 25

c) 26

d) 27

e) 28

17. If

$$\sin x + \cos x = \frac{1}{2}$$

what is $\sin^3 x + \cos^3 x$?

a) $\frac{11}{16}$

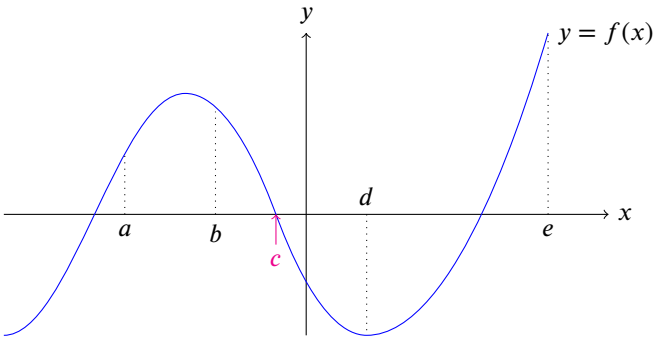
b) $\frac{1}{8}$

c) -1

d) $\frac{1}{3}$

e) 0

18. Consider the graph of a function $f : \mathbb{R} \to \mathbb{R}$ below.



i) $\frac{f(b)-f(a)}{b-a} > 0$

ii) $f(b)f(d) > 0$

iii) $f(c) > c$

iv) $f\left(\frac{d+e}{2}\right) > \frac{f(d)+f(e)}{2}$

Choose all the correct statements.

a) None of them

b) i), iii)

c) i), iv)

d) i), iii), iv)

e) All of them

19. If a certain polynomial is divided by $x - 1$, the remainder is 2. If the same polynomial is divided by $x - 2$, the remainder is 1. What is the remainder if the polynomial is divided by $(x - 1)(x - 2)$?

a) 3

b) $x + 1$

c) $2x + 1$

d) $x + 2$

e) $-x + 3$.

20. For nine numbers $a, b, c, d, e, f, g, h, i$ arranged in rectangular form

$$\begin{array}{cccc} a & b & c & \rightarrow \quad a+b+c=6 \\ d & e & f & \rightarrow \quad d+e+f=4 \\ g & h & i & \rightarrow \quad g+h+i=? \\ \downarrow & \downarrow & \downarrow & \\ a+d+g=3 & b+e+h=6 & c+f+i=13 & \end{array}$$

Homer Simpson  plans to compute the sums of rows and columns of the numbers. However, he is busy eating donuts and forgets to compute the sum of numbers in the 3rd row. Help him compute $g+h+i$.

a) $g+h+i=1$

b) $g+h+i=5$

c) $g+h+i=6$

d) $g+h+i=12$

e) $g+h+i=15$

21. Six people sat down along one side of a banquet table completely ignoring their name cards. In how many ways could this have been done so that no person was seated where his or her name card was placed?

a) 263

b) 265

c) 225

d) 235

e) 253

22. If the sequence $R_n = \frac{1}{2}(a^n + b^n)$, where $n \geq 0$, $a = 3 + 2\sqrt{2}$ and $b = 3 - 2\sqrt{2}$, find the unit digit of R_{12345}.

a) 7

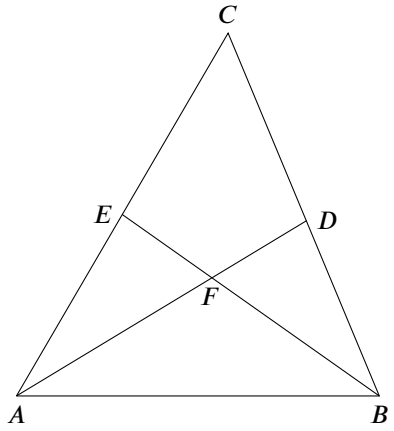
b) 5

c) 3

d) 9

e) 1

23. In $\triangle ABC$, $AB = 6$, $BC = 7$ and $CA = 8$. A point D lies on $\overline{BC}$ and $\overline{AD}$ bisects $\angle BAC$. Another point E lies on $\overline{AC}$ and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



a) $AF : FD = 3 : 2$

b) $AF : FD = 5 : 3$

c) $AF : FD = 2 : 1$

d) $AF : FD = 7 : 3$

e) $AF : FD = 5 : 2$

24. What is the sum of all positive integers less than 100 which have exactly 12 divisors?

a) 400

b) 402

c) 386

d) 406

e) 256

25. Suppose that $ABCD$ is a quadrilateral so that:

- The vertices A, B, C and D are all points on a circle.
- $AB = 5$, $BC = 12$.
- $ABCD$ is a parallelogram.

What is the length of AC?

a) 13

b) 14

c) $10\sqrt{2}$

d) $7\sqrt{5}$

e) $6\sqrt{10}$

26. How many integer values of k are there which allow us to factor $x^2 + kx - 18$ into $(x - a)(x - b)$ where a and b are integers?

a) 2

b) 3

c) 4

d) 6

e) 9

27. Which is the only of the following statements that can be true?

a) All five statements in this list are false.

b) Exactly four of the statements in this list are false.

c) Exactly three of the statements in this list are false.

d) Exactly two of the statements in this list are false.

e) Exactly one of the statements in this list is false.

28. Suppose that $|x + y| + |x - y| = 2$. What is the maximum possible value of $x^2 - 6x + y^2$?

a) 6

b) 9

c) 8

d) 7

e) 10

29. Let S be the set of all positive integers none of whose prime divisors is larger than 3. Thus

$$S = \{1, 2, 3, 4, 6, 8, 9, 12, ...\}$$

(For example, $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$ are in S but 5 and 7 are not in S.) What is the sum of reciprocals of the elements of S? In other words, what is the value of the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots ?$$

a) 2

b) 2.5

c) 3

d) 3.25

e) 3.5

30. If $f(x)$ satisfies

$$2f(x) + f\left(\frac{\pi}{2} - x\right) = \sin x \qquad \text{for all } x$$

then what is $f(x)$?

a) $f(x) = \sin x$

b) $f(x) = \cos x$

c) $f(x) = \frac{2\sin x - \cos x}{3}$

d) $f(x) = \frac{3\sin x + 4\cos x}{5}$

e) $f(x) = \frac{4\sin x + 5\cos x}{6}$

(∗) There is one "tie breaker problem" on the answer sheet.