The 48th Annual High School Mathematics Contest
April 13, 2022
Minnesotata State University, Mankato
In memory of Dr. Waters who loved math and the math contest.
Answers and Solutions

Instruction for the Contest

1. Write your answer on the **answer sheet**.

2. You have 90 minutes to work for on questions and one tie breaker problem.

3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.

4. You need to write a solution for the tie breaker problem on the **answer sheet**, and the tie breaker will be used only to break any possible ties that arise on the test.

5. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

6. **Submit your answer sheet** at the end of the contest and keep the exam booklet.

1. Two integer numbers between 1 and 9, inclusive, are selected at random. The same number may be selected twice. What is the probability that their product is a multiple of 3?

a) $36/81$

b) $5/9$

c) $5/8$

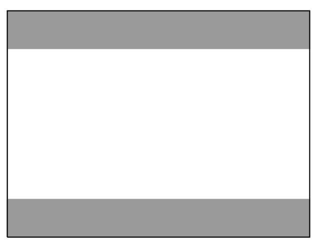
d) $25/81$

e) $20/81$

Answer: b)

Solution) Two numbers from $\{1,2,3,4,5,6,7,8,9\}$ may be chosen in $9 \times 9 = 81$ ways. If the product of the two numbers is not a multiple of 3, then each number in the product is not a multiple of 3 either, and so it must be chosen from $\{1,2,4,5,7,8\}$. This occurs in $6 \times 6 = 36$ ways. So there are $81 - 36 = 45$ ways to pick two numbers whose product is a multiple of 3. Thus the probability is $\frac{45}{81} = \frac{5}{9}$.

2. Older television screens have an aspect ratio of $4 : 3$. That is, the ratio of the width to the height is $4 : 3$. The aspect ratio of many movies is not $4 : 3$, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown below.



Suppose a movie has an aspect ratio of $2 : 1$ and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?

a) 2

b) 2.7

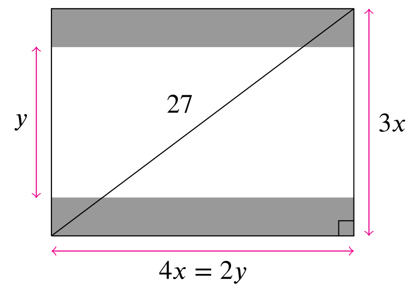
c) 2.5

d) 2.25

e) 3

Answer: b)

Solution) Let the width and height of the screen be $4x$ and $3x$ respectively, and let the width and height of the movie zone be $2y$ and y respectively.



By the Pythagorean Theorem

$$\sqrt{(3x)^2 + (4x)^2} = 27 \implies 5x = 7 \implies x = \frac{27}{5}.$$

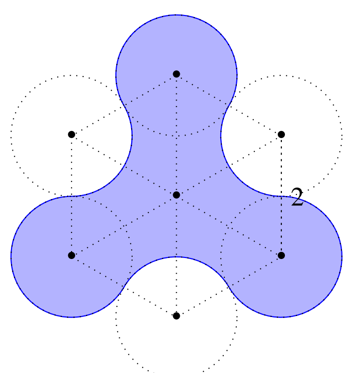
Since the movie zone and the screen have the same width, we have

$$2y = 4x \implies y = 2x.$$

and so the height of each strip is

$$\frac{3x - y}{2} = \frac{3x - 2x}{2} = \frac{x}{2} = \frac{27}{10} = 2.7.$$

3. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is on a vertex of a regular hexagon of side length 2.



What is the area enclosed by the curve?

a) $2\pi + 6$

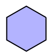
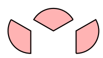
b) $2\pi + 4\sqrt{3}$

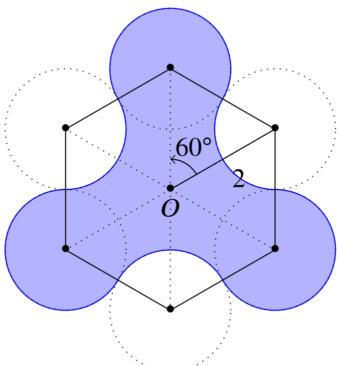
c) $3\pi + 4$

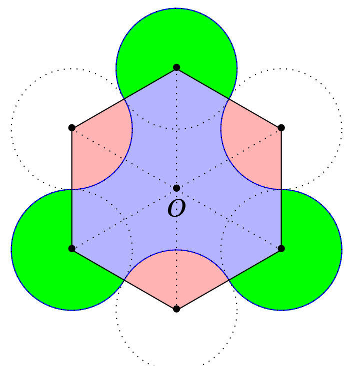
d) $2\pi + 2\sqrt{3} + 2$

e) $\pi + 6\sqrt{3}$

Answer: e)

Solution) The radius of the cirlces is 1. First, compute the area $6 \cdot \frac{1}{2} \cdot (2\sqrt{3}) = 6\sqrt{3}$ of the hexagon  between the centers of the circles. Then add the sum of areas $3 \cdot \frac{2}{3}\pi = 2\pi$ of the three sectors  outside the hexagon and subtract the sum $3 \cdot \frac{1}{3}\pi = \pi$ of areas of the three sectors  inside the hexagon and outside the closed curve.





The area enclosed by the curved figure is

$$6\sqrt{3} + 2\pi - \pi = \pi + 6\sqrt{3}.$$

4. Alice has taken 4 tests in her mathematics class. If her score on the next test is 100, then her average after taking all five tests will be 90. What is her average score *right now* (only considering the first four tests)?

a) 84.8

b) 85

c) 86.25

d) 87.5

e) 89.75

Answer: d)

Solution) Let T be her total score after the first four exams. Then

$$\frac{T + 100}{5} = 90 \implies T + 100 = 450 \implies T = 350.$$

Therefore she scored a total of 350 points on the first four exams, and so her average on those tests is $350/4 = 87.5$.

5. How many positive integers n are there so that $1! + 2! + 3! + \cdots + n! \leq n^3$?

(The factorial $n!$ means the product of the first n positive integers, that is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2) \cdot (n-1) \cdot n. \qquad)$$

a) 1

b) 2

c) 3

d) 4

e) 5

Answer: d)

Solution) Compute the first few values for n:

- $n = 1$: $1! \leq 1^3$
- $n = 2$: $1! + 2! = 3 \leq 8 = 2^3$
- $n = 3$: $1! + 2! + 3! = 9 \leq 27 = 3^3$
- $n = 4$: $1! + 2! + 3! + 4! = 33 \leq 64 = 4^3$

Since $5! = 120$ we get $1! + 2! + 3! + 4! + 5! = 33 + 120 = 153$, which is larger than $5^3 = 125$. From that point on, the sum of factorials grows at a greater rate than n^3 and consequently, $1! + 2! + 3! + \cdots + n! \geq n^3$ for all $n \geq 5$.

Remark) In fact,

$$(*) \quad n! \geq n^3 \quad \text{for all } n \geq 6 \implies 1! + 2! + 3! + \cdots + n! \geq n^3 \quad \text{for all } n \geq 6.$$

To prove $(*)$, we can use mathematical induction on n. The base case $n = 6$ is true. Assume the assertion is true for n. Then it is true for $n + 1$ because

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ &\geq (n+1) \cdot n^3 \geq (n+1)^3.\end{aligned}$$

6. Suppose that $a, b, 20, c, d$ is an arithmetic sequence. What is the value of $a + b + c + d$?

a.) 40

b.) 60

c.) 70

d.) 80

e.) 90

Answer: d)

Solution I) Let n be the common difference so that $b = a + n$, $20 = a + 2n$, $c = a + 3n$, $d = a + 4n$. Then

$$20 = a + 2n \implies n = 10 - \frac{a}{2}$$

and

$$\begin{aligned} a + b + c + d &= a + \left(a + 10 - \frac{a}{2}\right) + \left(a + 30 - \frac{3a}{2}\right) + \left(a + 40 - \frac{4a}{2}\right) \\ &= 80. \end{aligned}$$

Solution II) In an arithmetic sequence of five terms, the middle term is always the same as the average (since the other terms are equally spaced above and below it.) Therefore:

$$\frac{a + b + 20 + c + d}{5} = 20 \implies a + b + c + d = 80.$$

7. Bob goes onto a game show where the contestant gains 10 dollars for every right answer, but loses 6 dollars for every wrong answer. Bob answered 32 questions and left with 0 dollars. How many questions did he answer correctly?

a) 6

b) 10

c) 12

d) 13

e) 15

Answer: c)

Solution I) Let c be the number of correct answers and w the number of wrong answers. Then

$$\begin{cases} 10c - 6w = 0 \\ c + w = 32 \end{cases} \implies \begin{cases} c = 12 \\ w = 20 \end{cases}$$

Solution II) To break even Bob, must have

$$10c - 6w = 0 \implies w = \frac{5c}{3}$$

Since the number of wrong answers is an integer, the number of correct answers must be a multiple of 3.

- $c = 3 \implies w = \frac{5 \cdot 3}{3} = 5$ and $c + w = 8$.
- $c = 6 \implies w = \frac{5 \cdot 6}{3} = 10$ and $c + w = 16$.
- $c = 9 \implies w = \frac{5 \cdot 9}{3} = 15$ and $c + w = 24$.

- $c = 12 \implies w = \frac{5 \cdot 12}{3} = 20$ and $c + w = 32$.
- $c = 15 \implies w = \frac{5 \cdot 12}{3} = 25$ and $c + w = 40$.

Going through the multiples of 3 we can quickly find that $c = 12$ and $w = 20$ is the only way to have 32 questions in total.

8. Find the sum

$$\frac{1}{\sqrt{9}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{4}}.$$

a) 2

b) 3

c) 5

d) 1

e) $\sqrt{5} - \sqrt{2}$

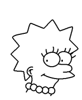
Answer: d)

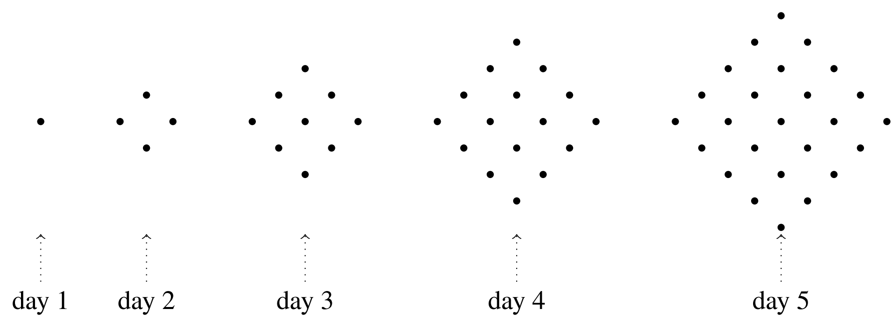
Solution) Let S be the sum. Rationalizing the denominator

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})} = \sqrt{n+1}-\sqrt{n} \qquad \text{for all } n \geq 1$$

we can write the expression as an alternating sum

$$\begin{aligned} S &= \frac{1}{\sqrt{9}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{4}} \\ &= (\sqrt{9}-\sqrt{8}) + (\sqrt{8}-\sqrt{7}) + (\sqrt{7}-\sqrt{6}) + (\sqrt{6}-\sqrt{5}) + (\sqrt{5}-\sqrt{4}) \\ &= \sqrt{9}-\sqrt{4} = 3-2 \\ &= 1. \end{aligned}$$

9. On the first day Lisa  draws 1 dot, on day 2 she draws 4 dots, on day 3 she draws 9 dots and she continues drawing dots in this way.



On day 17, how many dots does she draw?

a) 144

b) 215

c) 289

d) 342

e) 437

Answer: c).

Solution) Thinking of the pattern

$$\begin{array}{ccc}
\text{day } 1 & \to & 1^2 \text{ dots} \\
\text{day } 2 & \to & 2^2 \text{ dots} \\
\text{day } 3 & \to & 3^2 \text{ dots} \\
\text{day } 4 & \to & 4^2 \text{ dots} \\
\text{day } 5 & \to & 5^2 \text{ dots} \\
 & \vdots & \\
\text{day } 17 & \to & 17^2 \text{ dots}
\end{array}$$

we find that she draws $17^2 = 289$ dots on the 17th day.

10. If $f(x) = \frac{2x+3}{5x-7}$ then the inverse function $f^{-1}(x)$ of $f(x)$ is

a) $f^{-1}(x) = \frac{3x+5}{7x-2}$

b) $f^{-1}(x) = \frac{2x-7}{5x+3}$

c) $f^{-1}(x) = \frac{5x+2}{3x-7}$

d) $f^{-1}(x) = \frac{7x+3}{5x-2}$

e) $f^{-1}(x) = \frac{5x-7}{2x+3}$

Answer: d)

Solution) Let $y = \frac{2x+3}{5x-7}$ and solve for x:

$$y = \frac{2x+3}{5x-7} \implies y(5x-7) = 2x+3 \implies (5y-2)x = 7y+3 \implies x = \frac{7y+3}{5y-7}$$

The inverse function is $f^{-1}(y) = \frac{7y+3}{5y-7}$ or $f^{-1}(x) = \frac{7x+3}{5x-2}$ after changing the name of variable from y to x.

11. Compute the number of ways in which 20 one-dollar bills can be distributed to 4 people so that no person receives less than 4.

a) 35

b) 24

c) 15

d) 30

e) 10

Answer: a)

Solution I) Suppose the i-th person gets a_i dollars for $i = 1, \cdots, 4$. The problem is equivalent to counting the number of integer solutions satisfying

$$a_1 + a_2 + a_3 + a_4 = 20 \quad \text{where} \quad a_i \geq 4.$$

Using $b_i = a_i - 3$ it is equivalent to counting the number of integer solutions satisfying

$$(*) \quad b_1 + b_2 + b_3 + b_4 = 8 \quad \text{where } b_i \geq 1.$$

The number of solutions to $(*)$ is the same as the number of ways of putting 3 dividers in the 7 gaps of 8 objects in a line.

$$\underbrace{\square\ \ \square\ \ \square}_{b_1=3}\ \ |\ \ \underbrace{\square\ \ \square}_{b_2=2}\ \ |\ \ \underbrace{\ \square\ }_{b_3=1}\ \ |\ \ \underbrace{\square\ \ \square}_{b_4=2} \implies b_1 + b_2 + b_3 + b_4 = 8.$$

Therefore, $\binom{7}{3} = \frac{7!}{3!4!} = 35$ ways to distribute the bills.

Solution II) Let a_i be the sam above. Using $c_i = a_i - 4$, we reduce the problem of counting the number of integer solutions satisfying

$$c_1 + c_2 + c_3 + c_4 = 4 \qquad \text{where } c_i \geq 0.$$

There are 5 solutions satisfying $c_1 \geq c_2 \geq c_3 \geq c_4$ (this is the same as ignoring the ordering of c_1, c_2, c_3, c_4).

i) (4,0,0,0): There are $\binom{4}{1} = 4$ ways to distribute these amounts to c_1, c_2, c_3, c_4:
$$(c_1, c_2, c_3, c_4) = (4, 0, 0, 0), (0, 4, 0, 0), (0, 0, 4, 0), (0, 0, 0, 4).$$

ii) (3,1,0,0): There are $\binom{4}{2} = 12$ ways to distribute these amounts to c_1, c_2, c_3, c_4:
$$\begin{aligned}(c_1, c_2, c_3, c_4) = {} & (3, 1, 0, 0), (3, 0, 1, 0), (3, 0, 0, 1), (0, 3, 1, 0), (0, 3, 0, 1), (0, 0, 3, 1) \\ & (1, 3, 0, 0), (1, 0, 3, 0), (1, 0, 0, 3), (0, 1, 3, 0), (0, 1, 0, 3), (0, 0, 1, 3)\end{aligned}$$

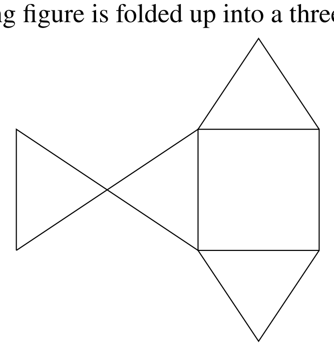
iii) (2,2,0,0): There are $\frac{\binom{4}{3}}{2!} = 6$ ways to distribute these amounts to c_1, c_2, c_3, c_4:
$$(c_1, c_2, c_3, c_4) = (2, 2, 0, 0), (2, 0, 2, 0), (2, 0, 0, 2), (0, 2, 2, 0), (0, 2, 0, 2), (0, 0, 2, 2)$$

iv) (2,1,1,0): There are $\binom{4}{2} = 12$ ways to distribute these amounts to c_1, c_2, c_3, c_4:
$$\begin{aligned}(c_1, c_2, c_3, c_4) = {} & (2, 0, 1, 1), (2, 1, 0, 1), (2, 1, 1, 0), (1, 2, 0, 1), (1, 2, 1, 0), (1, 1, 2, 0) \\ & (0, 2, 1, 1), (0, 1, 2, 1), (0, 1, 1, 2), (1, 0, 2, 1), (1, 0, 1, 2), (1, 1, 0, 2)\end{aligned}$$

v) (1,1,1,1): There is $\binom{4}{4} = 1$ way to distribute these amounts to c_1, c_2, c_3, c_4:
$$(c_1, c_2, c_3, c_4) = (1, 1, 1, 1).$$

There are $4 + 12 + 6 + 12 + 1 = 35$ ways to distribute the bills.

12. Suppose that the following figure is folded up into a three dimensional object



What sort of solid will it make?

a) Tetrahedron (Triangular pyramid)

b) Rectangular prism (Box)

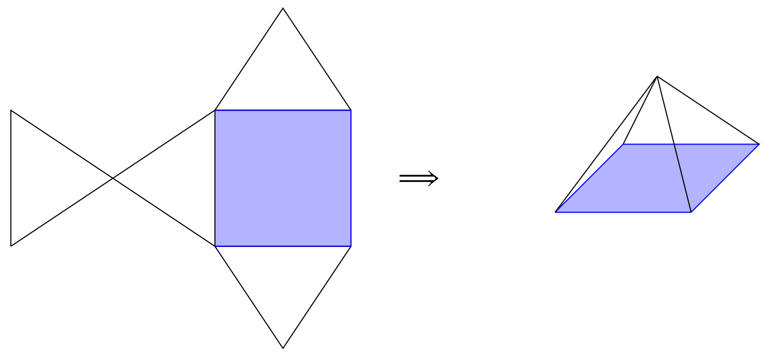
c) Triangular prism

d) Square pyramid

e) Cone

Answer: d)

Solution I) Thinking of the base of the 3 dimensional object (retangle), we find that the folded one is a square pyramid.



Solution II) The problem can be solved by process of elimination: an unfolded tetrahedron or cone would not have square component, an unfolded box would not have a triangular component. In a triangular prism there would be three rectangular components, not only one.

13. The equation $x^2 + 2x + y^2 + 6y + 6 = 0$ defines a circle. What is its radius?

a) 2

b) $\sqrt{6}$

c) $\sqrt{14}$

d) 4

e) None of the above

Answer: a)

Solution) Completing the squares of the left hand side of the equation

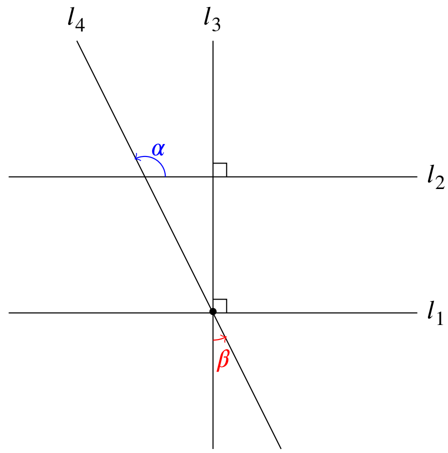
$$\begin{aligned} x^2 + 2x + y^2 + 6y + 6 &= x^2 + 2x + 1 - 1 + y^2 + 6y + 9 - 9 + 6 \\ &= (x+1)^2 + (y+3)^2 - 9 - 4 \end{aligned}$$

we find that

$$x^2 + 2x + y^2 + 6y + 6 = 0 \implies (x+1)^2 + (y+3)^2 = 2^2.$$

The radius is 2 and the center is $(-1, -3)$.

14. Consider the following picture



The line l_1 is parallel to the line l_2, and the line l_3 is perpendicular to the line l_1. The angle $\angle\alpha$ measures $110°$ (the picture is not to scale). What is the measure of $\angle\beta$?

a) $20°$

b) $30°$

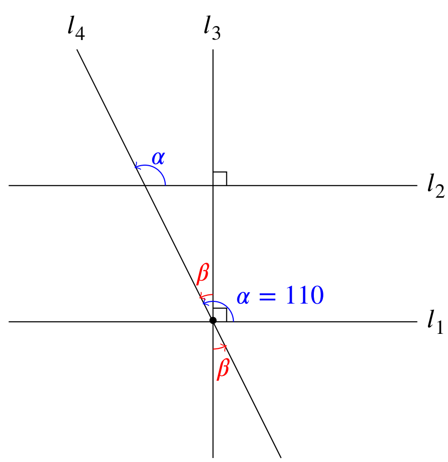
c) $35°$

d) $40°$

e) $45°$

Answer: a)

Solution) Since l_1 and l_2 are parallel, the larger angle made by l_4 and l_2 corresponds to $\angle\alpha$ and also measures $\alpha = 110°$. Since the measure between l_3 and l_1 is $90°$, this implies that the smaller angle between l_4 and l_1 measures $110° - 90° = 20°$.



15. A teacher with a math class of 20 students randomly pairs the students to take a test. What is the probability that Camilla and Cameron, two students in the class, are paired with

each other?

a) $3/25$

b) $1/24$

c) $1/18$

d) $1/20$

e) $1/19$

Answer: e)

Solution I) If the pairing is done at random, Camilla is equally likely to be paired with all 19 other students, so that the probability of being paired with Cameron is $\frac{1}{19}$.

Solution II) There are

$$\underbrace{(\square,\square)(\square,\square)\cdots(\square,\square)(\square,\square)}_{\text{10 pairs}} \Longrightarrow \frac{\binom{20}{2}\binom{18}{2}\cdots\binom{4}{2}\binom{2}{2}}{10!}$$

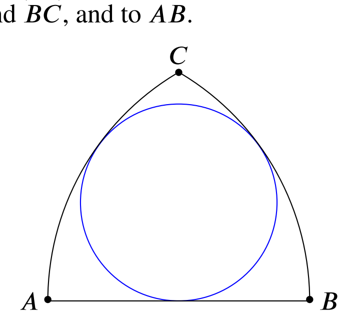
ways to form 10 pairs out of 20 students. If Camilla and Cameron must be paired, then there are

$$\underbrace{(\square,\square)}_{\text{Camilla and Cameron}} \quad \underbrace{(\square,\square)\cdots(\square,\square)(\square,\square)}_{\text{9 pairs}} \Longrightarrow \frac{\binom{18}{2}\cdots\binom{4}{2}\binom{2}{2}}{9!}$$

ways to form 9 pairs out of 18 students. The probability that Camilla and Cameron are paired with each other is

$$\frac{\frac{\binom{18}{2}\cdots\binom{4}{2}\binom{2}{2}}{9!}}{\frac{\binom{20}{2}\binom{18}{2}\cdots\binom{4}{2}\binom{2}{2}}{10!}} = \frac{10}{\binom{20}{2}} = \frac{1}{19}.$$

16. If circular arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ have centers at B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$.



If the length of $\overset{\frown}{BC}$ is 12, then the circumference of the circle is

a) 24

b) 25

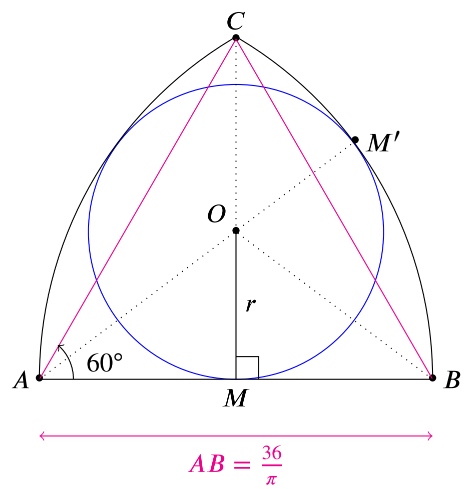
c) 26

d) 27

e) 28

Answer: d)

Solution) Note the triangle ABC is equilateral because AB, AC, AC is the common radius, and so $\angle CAB = 60°$.



Considering the circular sector with arc $\overset{\frown}{BC}$ centered at A of radius AB, we find that

$$\frac{1}{6}(2\pi)AB = 12 \implies AB = \frac{36}{\pi}.$$

Draw a perpendicular from the center O of the circle to the side AB, and call this length r, and call the foot M. Draw an extended line from A to O which meets the circular arc $\overset{\frown}{BC}$ tangentially at M' and

$$\frac{36}{\pi} = AM' = OA + OM' \text{ and } OM' = r \implies OA = \frac{36}{\pi} - r.$$

Applying the Pythagorean Theorem to $\triangle AMO$, we find that

$$r^2 + \left(\frac{18}{\pi}\right)^2 = \left(\frac{36}{\pi} - r\right)^2 \implies r = \frac{27}{2\pi}$$

Finally, we see that the circumference is $2\pi \cdot 27/2\pi = 27$.

17. If

$$\sin x + \cos x = \frac{1}{2}$$

what is $\sin^3 x + \cos^3 x$?

a) $\frac{11}{16}$

b) $\frac{1}{8}$

c) -1

d) $\frac{1}{3}$

e) 0

Answer: a)

Solution) Using the identity $\sin^2 x + \cos^2 x = 1$, we find that

$$(\sin x + \cos x)^2 = \left(\frac{1}{2}\right)^2 \implies 1 + 2\sin x \cos x = \frac{1}{4} \implies \sin x \cos x = -\frac{3}{8}$$

and

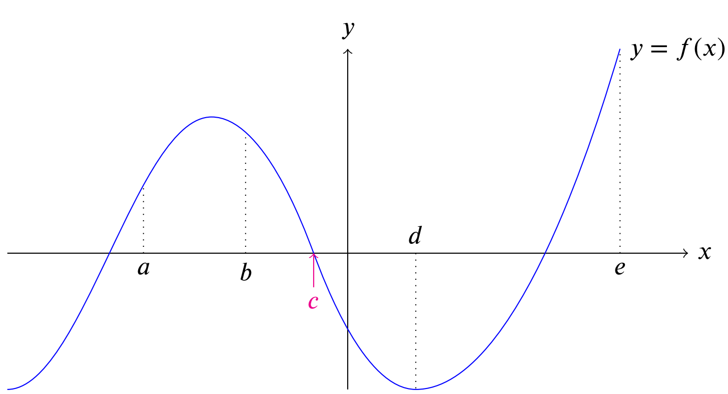
$$\begin{aligned}\sin^3 x + \cos^3 x &= (\sin x + \cos x)^3 - 3\sin x \cos x(\sin x + \cos x)\\ &= \left(\frac{1}{2}\right)^3 - 3\cdot\left(-\frac{3}{8}\right)\cdot\frac{1}{2} = \frac{11}{16}\end{aligned}$$

Remark) The identity

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) \qquad \text{for all } a, b$$

is used above.

18. Consider the graph of a function $f : \mathbb{R} \to \mathbb{R}$ below.



i) $\frac{f(b)-f(a)}{b-a} > 0$

ii) $f(b)f(d) > 0$

iii) $f(c) > c$

iv) $f\left(\frac{d+e}{2}\right) > \frac{f(d)+f(e)}{2}$

Choose all the correct statements.

a) None of them

b) i), iii)

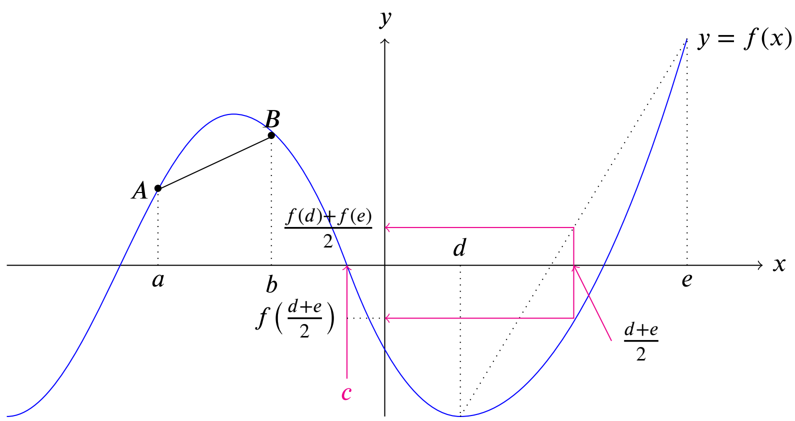
c) i), iv)

d) i), iii), iv)

e) All of them

Answer: b)

Solution) Look at the graph of f below



i) is true becaues $\frac{f(b)-f(a)}{b-a}$ is the slope of the line AB which is positive.

ii) is false because $f(b) > 0$ and $f(d) < 0$ and consequently $f(b)f(d) < 0$

iii) is true because $f(c) = 0$ and $c < 0$

iv) is false because $f\left(\frac{d+e}{2}\right) < \frac{f(d)+f(e)}{2}$ (see the figure above).

19. If a certain polynomial is divided by $x-1$, the remainder is 2. If the same polynomial is divided by $x-2$, the remainder is 1. What is the remainder if the polynomial is divided by $(x-1)(x-2)$?

a) 3

b) $x+1$

c) $2x+1$

d) $x+2$

e) $-x+3$.

Answer: e).

Solution I) Let $f(x)$ be the polynomial in question. By the division algorithm for polynomial, we know that there are polynomials $q_1(x)$ and $q_2(x)$ such that

$$\begin{cases} f(x) = (x-1)q_1(x) + 2 & \implies f(1) = 2 \\ f(x) = (x-2)q_2(x) + 1 & \implies f(2) = 1 \end{cases}$$

Applying the division algorithm again, we find that there is a polynomial $q(x)$ such that

$$f(x) = (x-1)(x-2)q(x) + ax + b$$

where a, b are numbers. To find a, b, we use $f(1) = 2$ and $f(2) = 1$

$$\begin{cases} f(x) = (x-1)(x-2)q(x) + ax + b \text{ (plug in } x = 1) & \implies a + b = 2 \\ f(x) = (x-1)(x-2)q(x) + ax + b \text{ (plug in } x = 2) & \implies 2a + b = 1 \end{cases} \implies \begin{cases} a = -1 \\ b = 3 \end{cases}$$

and the remainder is $-x+3$.

Solution II) Another way of solving the problem is to come up with a polynomial $f(x)$ with required property that $f(1) = 2$ and $f(2) = 1$. For example, $f(x) = x^2 - 4x + 5$ satisfies $f(1) = 2$ and $f(2) = 1$. Dividing $f(x) = x^2 - 4x + 5$ by $(x-1)(x-2) = x^2 - 3x + 2$

$$x^2 - 4x + 5 = (x^2 - 3x + 2) \cdot 1 + (-x + 3)$$

we find the remainder $-x+3$.

Remark) The division algorithm for polynomial states that if a polynomial $f(x)$ is divided by another polynomial $g(x)$, there exists polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x) \qquad \text{and} \qquad \deg r(x) \leq \deg g(x).$$

20. For nine numbers $a, b, c, d, e, f, g, h, i$ arranged in rectangular form

$$\begin{array}{ccccc} a & b & c & \rightarrow & a+b+c=6 \\ d & e & f & \rightarrow & d+e+f=4 \\ g & h & i & \rightarrow & g+h+i=? \\ \downarrow & \downarrow & \downarrow & & \\ a+d+g=3 & b+e+h=6 & c+f+i=13 & & \end{array}$$

Homer Simpson  plans to compute the sums of rows and columns of the numbers. However, he is busy eating donuts and forgets to compute the sum of numbers in the 3rd row. Help him compute $g + h + i$.

a) $g + h + i = 1$

b) $g + h + i = 5$

c) $g + h + i = 6$

d) $g + h + i = 12$

e) $g + h + i = 15$

Answer: d).

Solution) The sum of the sums of columns is the same as the sum of the sum of rows and so

$$\begin{aligned} g + h + i &= [(a + d + g) + (b + e + h) + (c + f + i)] - [(a + b + c) + (d + e + f)] \\ &= (3 + 6 + 13) - (6 + 4) = 12. \end{aligned}$$

Remark) The problem is not void as the following array of numbers shows

$$\begin{array}{ccccc} 1 & 2 & 3 & \rightarrow & 1 + 2 + 3 = 6 \\ 1 & -1 & 4 & \rightarrow & 1 + (-1) + 4 = 4 \\ 1 & 5 & 6 & \rightarrow & 1 + 5 + 6 = 12 \\ \downarrow & \downarrow & \downarrow & & \\ 1 + 1 + 1 = 3 & 2 + (-1) + 5 = 6 & 3 + 4 + 6 = 13 & & \end{array}$$

21. Six people sat down along one side of a banquet table completely ignoring their name cards. In how many ways could this have been done so that no person was seated where his or her name card was placed?

a) 263

b) 265

c) 225

d) 235

e) 253

Answer: b)

Solution) Let S be the set of all possible seating arrangements and let S_i be the set of seating arrangements that the i-th person is seated where his or her name card was placed. Then the number of seating arrangements that everyone seats in the wrong place is the number of elements in the set $S-(S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_5)$. We shall count the number of elements in $S - \cup_{i=1}^{k} S_i$ by using the principle of inclusion and exclusion

$$|S-\bigcup_{i=1}^{k} S_i| = |S|-(|S_1|+\cdots+|S_6|)+(|S_1\cap S_2|+\cdots+|S_5\cap S_6|)-\cdots+|S_1\cap S_2\cap S_3\cap S_4\cap S_5\cap S_6|$$

There are $|S| = 6!$ many elements in S

$$S=\left\{\begin{pmatrix}\text{name card:} & 1 & 2 & 3 & 4 & 5 & 6\\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ \text{actual seating:} & \square & \square & \square & \square & \square & \square\end{pmatrix}\right\} \implies |S| = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 6!$$

There are $|S_i| = 5!$ many elements in S_i and there are $\binom{6}{1}$ many such sets (namely, $S_1, S_2, ..., S_6$), for example,

$$S_1=\left\{\begin{pmatrix}\text{name card:} & 1 & 2 & 3 & 4 & 5 & 6\\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ \text{actual seating:} & 1 & \square & \square & \square & \square & \square\end{pmatrix}\right\} \implies |S_1| = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5!$$

For each $i < j$, there are $|S_i \cap S_j| = 4!$ many elements in $S_i \cap S_j$, and there are $\binom{6}{2}$ many such sets (namely, $S_1\cap S_2, S_1\cap S_3, ..., S_5\cap S_6$), for example

$$S_1\cap S_2=\left\{\begin{pmatrix}\text{name card:} & 1 & 2 & 3 & 4 & 5 & 6\\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ \text{actual seating:} & 1 & 2 & \square & \square & \square & \square\end{pmatrix}\right\} \implies |S_1\cap S_2| = 4\cdot 3\cdot 2\cdot 1 = 4!$$

Continuing in this fashion and using the principle of inclusion and exclusion, we find that

$$\begin{aligned}|S-\bigcup_{i=1}^{k} S_i| &= |S| - \sum_{i=1}^{6}|S_i| + \sum_{i<j}^{6}|S_i\cap S_j| - \cdots + |S_1\cap S_2\cap S_3\cap S_4\cap S_5\cap S_6|\\ &= 6! - \binom{6}{1}5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5}1! + \binom{6}{6}0!\\ &= 265.\end{aligned}$$

22. If the sequence $R_n = \frac{1}{2}(a^n + b^n)$, where $n \geq 0$, $a = 3+2\sqrt{2}$ and $b = 3-2\sqrt{2}$, find the unit digit of R_{12345}.

a) 7

b) 5

c) 3

d) 9

e) 1

Answer: d)

Solution) Since a, b are the roots of the equation

$$(*)\quad (\lambda - a)(\lambda - b) = (\lambda - (3 + 2\sqrt{2})(\lambda - (3 - 2\sqrt{2}) = \lambda^2 - 6\lambda + 1 = 0$$

R_n satisfies

$$\begin{aligned} R_{n+2} - 6R_{n+1} + R_n &= \frac{1}{2}[(a^{n+2} + b^{n+2}) - 6(a^{n+1} + b^{n+1}) + a^n + b^n] \\ &= \frac{1}{2}[a^n(a^2 - 6a + 1) + b^n(b^2 - 6b + 1)] = 0 \end{aligned}$$

We have the recurrence relation for R_n as follows

$$R_{n+2} = 6R_{n+1} - R_n \quad \text{for } n \geq 0 \quad \text{and} \qquad R_0 = 1, R_1 = 3.$$

Since we look for the unit digit of R_n, we consider the sequence R_n modulo 10

$$\begin{aligned} R_0 &= 1 & &\equiv 1 \mod 10 \\ R_1 &= 3 & &\equiv 3 \mod 10 \\ R_2 &= 6R_1 - R_0 &\equiv 6 \cdot 3 - 1 &\equiv 7 \mod 10 \\ R_3 &= 6R_2 - R_1 &\equiv 6 \cdot 7 - 3 &\equiv 9 \mod 10 \\ R_4 &= 6R_3 - R_2 &\equiv 6 \cdot 9 - 7 &\equiv 7 \mod 10 \\ R_5 &= 6R_4 - R_3 &\equiv 6 \cdot 7 - 9 &\equiv 3 \mod 10 \\ R_6 &= 6R_5 - R_4 &\equiv 6 \cdot 3 - 7 &\equiv 1 \mod 10 \\ R_7 &= 6R_6 - R_5 &\equiv 6 \cdot 1 - 3 &\equiv 3 \mod 10 \\ R_8 &= 6R_7 - R_6 &\equiv 6 \cdot 3 - 1 &\equiv 7 \mod 10 \\ &\vdots \end{aligned}$$

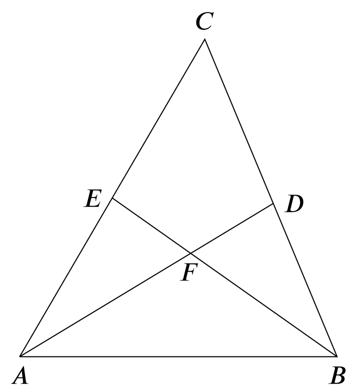
and find that the sequence R_n modulo 10 is periodic with period 6 because $R_6 \equiv R_0, R_7 \equiv R_1, R_8 \equiv R_2 \mod 10$ etc. Since $12345 \equiv 3 \mod 6$, we find that

$$R_{12345} \equiv R_3 \equiv 9 \mod 10$$

which means that the unit digit is 9.

Remark) The equation $(*)$ is called the characteristic equation for R_n.

23. In $\triangle ABC$, $AB = 6$, $BC = 7$ and $CA = 8$. A point D lies on $\overline{BC}$ and $\overline{AD}$ bisects $\angle BAC$. Another point E lies on $\overline{AC}$ and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



a) $AF : FD = 3 : 2$

b) $AF : FD = 5 : 3$

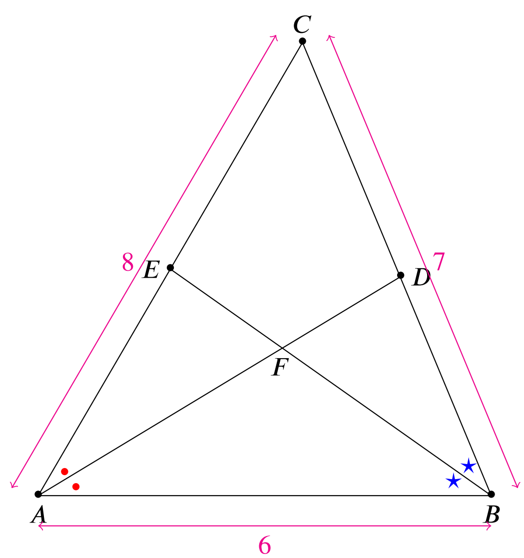
c) $AF : FD = 2 : 1$

d) $AF : FD = 7 : 3$

e) $AF : FD = 5 : 2$

Answer: c)

Solution) Applying the angle bisector theorem to $\triangle ABD$,



we find that

$$\frac{AF}{FD} = \frac{AB}{BD} \implies \frac{AF}{FD} = \frac{6}{BD}.$$

To find BD, we apply the angle bisector theorem to $\triangle BAC$

$$\frac{BD}{DC} = \frac{BA}{AC} \implies \frac{BD}{DC} = \frac{6}{8} \implies \begin{cases} BD : DC = 3 : 4 \\ BD + DC = 7 \end{cases} \implies BD = 3$$

We conclude that

$$AF : FD = AB : BD = 6 : 3 = 2 : 1$$

24. What is the sum of all positive integers less than 100 which have exactly 12 divisors?

a) 400

b) 402

c) 386

d) 406

e) 256

Answer: b)

Solution) Since the number of divisors of $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is $(e_1+1)(e_2+1)\cdots(e_k+1)$, where $p_1, ..., p_k$ are distinct primes and $e_1, ..., e_k \geq 1$, we look for positive integers $n \leq 100$ such that

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} < 100 \quad \text{with} \quad (e_1+1)(e_2+1)\cdots(e_k+1) = 12.$$

There are several cases to consider:

i) For $k = 1$, we consider integers of the form

$$n = p^a < 100 \quad \text{(prime } p \text{ and } a \geq 1) \text{ with } a + 1 = 12 \implies a = 11.$$

Since $n = p^{11} \geq 2^{11} > 100$ there is no such integer n.

ii) For $k = 2$, we consider integers of the form

$$n = p^a q^b < 100 \quad \text{(primes } p > q \text{ and } a, b \geq 1) \text{ with} \quad (a+1)(b+1) = 12$$

Then

$$(a+1)(b+1) = 12 \implies \begin{cases} a+1 = 2 \\ b+1 = 6 \end{cases} \text{ or } \begin{cases} a+1 = 3 \\ b+1 = 4 \end{cases} \text{ or } \begin{cases} a+1 = 6 \\ b+1 = 2 \end{cases}$$

and consequently there are three possible types of prime factorizations of n

$$\begin{cases} (a,b) = (1,5) \\ (a,b) = (2,3) \\ (a,b) = (5,1) \end{cases} \implies \begin{cases} n = pq^5 \\ n = p^2 q^3 \\ n = p^5 q \end{cases} \qquad (p > q)$$

If $p \geq 5$ and $q \geq 3$, then

$$n \geq \min\{p^2 q^3, pq^5\} \geq \min\{5^2 \cdot 3^3, 5 \cdot 3^5\} > 100$$

and we are left with the cases

$$(p,q)=(3,2) \implies \begin{cases} n = 3\cdot 2^5 = 96 < 100 \\ n = 3^2\cdot 2^3 = 72 < 100 \\ n = 3^5 \cdot 2^1 > 100 \end{cases}$$

iii) For $k=3$, we consider integers of the form

$n = p^a q^b r^c < 100$ (primes p,q,r and $a,b,c \geq 1$) with $(a+1)(b+1)(c+1) = 12$

Then

$$(a+1)(b+1)(c+1) = 12 \implies \begin{cases} a+1=2 \\ b+1=2 \\ c+1=3 \end{cases} \implies \begin{cases} a=1 \\ b=1 \\ c=2 \end{cases}$$

and consequently there is only one possible type of factorization $n = pqr^2$ with distinct primes p,q,r. If $r \geq 5$, then $n = pqr^2 \geq 2\cdot 3\cdot 5^2 > 100$. We are left with cases $r=2$ or $r=3$:

$$\begin{cases} (p,q,r) = (5,3,2) & \implies n = pqr^2 = 5\cdot 3\cdot 2^2 = 60 \\ (p,q,r) = (7,3,2) & \implies n = pqr^2 = 7\cdot 3\cdot 2^2 = 84 \\ (p,q,r) = (5,2,3) & \implies n = pqr^2 = 5\cdot 2\cdot 3^2 = 90 \end{cases}$$

and there are no more cases.

iv) If $k \geq 4$, then

$$(e_1+1)(e_2+1)\cdots(e_k+1) \geq (e_1+1)(e_2+1)(e_3+1)(e_4+1) \geq 2^4 > 12$$

and there are no such integers.

The sum of integers n satisfying the required conditions is

$$96 + 72 + 60 + 84 + 90 = 402.$$

25. Suppose that $ABCD$ is a quadrilateral so that:

- The vertices A, B, C and D are all points on a circle.
- $AB = 5$, $BC = 12$.
- $ABCD$ is a parallelogram.

What is the length of AC?

a) 13

b) 14

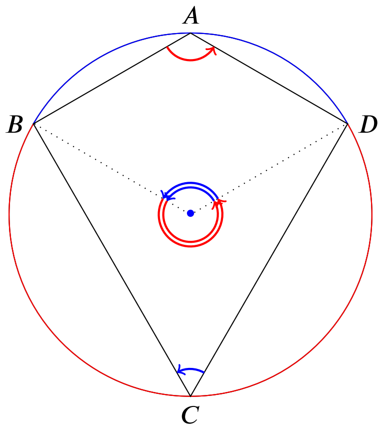
c) $10\sqrt{2}$

d) $7\sqrt{5}$

e) $6\sqrt{10}$

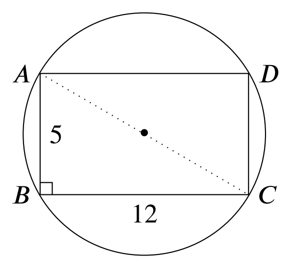
Answer: a)

Solution) Since the quadrilateral $ABCD$ is inscribed on a circle its opposite angles must be supplementary by the inscribed angle theorem



$$\implies \angle BAC + \angle BCD = 180°$$

Since $\square ABCD$ is a parallelogram, its opposite angles must also be 90°. This is only possible if $\square ABCD$ is a rectangle.



Applying the Pythagorean theorem to the right triangle $\triangle ABC$, we find that

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{5^2 + 12^2} = 13.$$

26. How many integer values of k are there which allow us to factor $x^2 + kx - 18$ into $(x - a)(x - b)$ where a and b are integers?

a) 2

b) 3

c) 4

d) 6

e) 9

Answer: d)

Solution) If k is such an integer, then we have

$$(x-a)(x-b) = x^2+kx-18 \implies x^2-(a+b)x+ab = x^2+kx-18 \implies \begin{cases} a+b=-k \\ ab=-18 \end{cases}$$

Thinking of all possible factorizations of -18 and symmetry on a, b for $k = -a - b$, we find that there are 6 such integers

a	1	2	3	6	9	18
b	-18	-9	-6	-3	-2	1
$k = -a - b$	-17	-7	-3	3	7	17

27. Which is the only of the following statements that can be true?

a) All five statements in this list are false.

b) Exactly four of the statements in this list are false.

c) Exactly three of the statements in this list are false.

d) Exactly two of the statements in this list are false.

e) Exactly one of the statements in this list is false.

Answer: b)

Solution) Exactly four statements are false. Any other statement being true leads to a contradiction. Let (T, F) be the exact numbers of true and false statements of each claim. Then

- a) claims $(\mathrm{T},\mathrm{F}) = (0,5)$
- b) claims $(\mathrm{T},\mathrm{F}) = (1,4)$
- c) claims $(\mathrm{T},\mathrm{F}) = (2,3)$
- d) claims $(\mathrm{T},\mathrm{F}) = (3,2)$
- e) claims $(\mathrm{T},\mathrm{F}) = (4,1)$

Now we analyze each claim:

- If a) were true it would contradict its own claim that it is false.
- b) can be true because it is possible that b) itself is true and the others are false.
- If c),d) or e) were true then there would be at least two true statements. However, every statement claims a different number of true statements and so there cannot be more than two true statements.

28. Suppose that $|x+y|+|x-y| = 2$. What is the maximum possible value of x^2-6x+y^2?

a) 6

b) 9

c) 8

d) 7

e) 10

Answer: c)

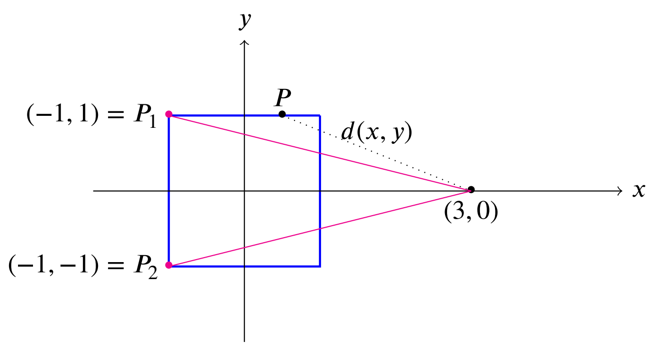
Solution) The points $P = (x, y)$ satisfying $|x + y| + |x - y| = 2$ form the boundary of the square with vertices $P_1 = (-1, 1)$, $P_2 = (-1, -1)$, $P_3 = (1, -1)$, $P_4 = (1, 1)$ because

$$\begin{cases} x + y \geq 0 \text{ and } x - y \geq 0 & \Longrightarrow \ |x + y| + |x - y| = 2x = 2 \quad \text{for } -x \leq y \leq x \\ x + y \geq 0 \text{ and } x - y \leq 0 & \Longrightarrow \ |x + y| + |x - y| = 2y = 2 \quad \text{for } -y \leq x \leq y \\ x + y \leq 0 \text{ and } x - y \geq 0 & \Longrightarrow \ |x + y| + |x - y| = -2y = 2 \quad \text{for } -y \leq x \leq y \\ x + y \leq 0 \text{ and } x - y \leq 0 & \Longrightarrow \ |x + y| + |x - y| = -2x = 2 \quad \text{for } -x \leq y \leq x \end{cases}$$

Completing the square, we write

$$f(x, y) = x^2 - 6x + y^2 = (x - 3)^2 + y^2 - 9 = d(x, y)^2 - 9$$

where $d(x, y) = \sqrt{(x - 3)^2 + y^2}$ is the distance between $P = (x, y)$ on the square and $(3, 0)$.



The value $f(x, y)$ maximizes when $d(x, y)$ maximizes. A simple geometric consideration tells us that the maximum distance $d_{\max}(x, y)$ occurs either at $P_1 = (-1, 1)$ or at $P_2 = (-1, -1)$ and $d_{\max} = \sqrt{4^2 + 1^2} = \sqrt{17}$. We find that

$$f_{\max} = d^2_{\max} - 9 = (\sqrt{17})^2 - 9 = 8.$$

29. Let S be the set of all positive integers none of whose prime divisors is larger than 3. Thus

$$S = \{1, 2, 3, 4, 6, 8, 9, 12, ...\}$$

(For example, $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$ are in S but 5 and 7 are not in S.) What is the sum of reciprocals of the elements of S? In other words, what is the value of the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots ?$$

a) 2

b) 2.5

c) 3

d) 3.25

e) 3.5

Answer: c)

Solution) Thinking of prime factorizations of integers in S, we find that

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots$$

For example, $\frac{1}{6}=\frac{1}{2}\cdot\frac{1}{3}$, and the all the terms $\frac{1}{2^m\cdot 3^n}$ in the sum are obtained in this way. By the geometric sum formula

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots=\frac{1}{1-\frac{1}{2}}=2 \quad \text{and} \quad 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

and so the required sum is $2\cdot\frac{3}{2}=3$.

30. If $f(x)$ satisfies

$$2f(x)+f\left(\frac{\pi}{2}-x\right)=\sin x \qquad \text{for all } x$$

then what is $f(x)$?

a) $f(x)=\sin x$

b) $f(x)=\cos x$

c) $f(x)=\frac{2\sin x-\cos x}{3}$

d) $f(x)=\frac{3\sin x+4\cos x}{5}$

e) $f(x)=\frac{4\sin x+5\cos x}{6}$

Answer: c)

Solution) Substituing x by $\frac{\pi}{2}-x$ in the functional equation we find that

$$2f\left(\frac{\pi}{2}-x\right)+f(x)=\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

and that

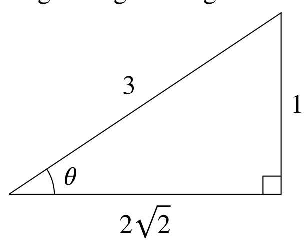
$$\begin{cases}2f(x)+f\left(\frac{\pi}{2}-x\right)=\sin x\\ 2f\left(\frac{\pi}{2}-x\right)+f(x)=\cos x\end{cases} \implies \begin{cases}4f(x)+2f\left(\frac{\pi}{2}-x\right)=2\sin x\\ 2f\left(\frac{\pi}{2}-x\right)+f(x)=\cos x\end{cases}$$

Subtracting the 2nd equation from the 1st, we obtain that

$$3f(x) = 2\sin x - \cos x \implies f(x) = \frac{2\sin x - \cos x}{3}.$$

Tie breaker problem: Find $\sum_{n=0}^{\infty} \frac{\sin(n\theta)}{3^n}$ if $\sin\theta = 1/3$ and $0 \le \theta \le \pi/2$.

Solution) Considering the right triangle below we find that



$$\implies \cos\theta = \frac{2\sqrt{2}}{3}$$

Let

$$z = e^{i\theta} = \cos\theta + i\sin\theta = \frac{2\sqrt{2}}{3} + \frac{1}{3}i \implies \sin\theta = \mathrm{Im}(z)$$

where $\mathrm{Im}(z)$ is the imaginary part of z. By Euler's formula,

$$z^n = e^{in\theta} = \cos(nx) + i\sin(nx) \implies \sin n\theta = \mathrm{Im}(z^n)$$

we have

$$\sum_{n=0}^{\infty} \frac{\sin n\theta}{3^n} = \mathrm{Im}\left(\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\right).$$

Since $\left|\frac{z}{3}\right| = \frac{1}{3} < 1$, the geometric series converges and its sum is

$$\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n = \frac{1}{1 - \frac{z}{3}} = \frac{9}{9 - 3z} = \frac{9}{9 - 2\sqrt{2} - i} = \frac{9\cdot(9 - 2\sqrt{2} + i)}{(9 - 2\sqrt{2})^2 + 1}$$

and we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sin n\theta}{3^n} &= \mathrm{Im}\left(\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\right) \\
&= \mathrm{Im}\left(\frac{9\cdot(9 - 2\sqrt{2} + i)}{(9 - 2\sqrt{2})^2 + 1}\right) \\
&= \frac{9}{(9 - 2\sqrt{2})^2 + 1} = \frac{5 + 2\sqrt{2}}{34}.
\end{aligned}$$