# Minnesotata State University, Mankato
# The 46th Annual High School Mathematics Contest
# April 24, 2019

1. You have 90 minutes to work for 30 questions on the test.

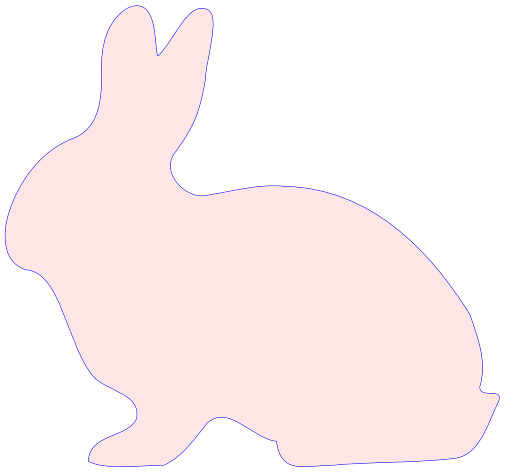
2. This is a multiple choice test. The choice **none of these** means not (a), (b), (c) or (d). Place your answer on the answer sheet in the corresponding space.

3. The questions on the last page of the test will be used by the judges to break any possible ties that arise on the test. Please put all of your work on the last page. Make sure your name and school code are at the top of the page.

4. The use of calculators during the exam is **NOT** permitted.

5. You may leave early if you finish the test early.

6. When you finish the test, keep the exam with you but sumbit the **answer sheet** with the last page of the booklet (the **tie breakers**).



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1. If $x + y = 7$ and $xy = 6$ then $\dfrac{1}{x} + \dfrac{1}{y}$ is

a) $\frac{6}{7}$

b) $\frac{7}{6}$

c) $\frac{12}{7}$

d) $\frac{12}{14}$

e) $\frac{18}{7}$

2. If $\dfrac{x-y}{x+y} = \dfrac{2018}{2019}$ then $\dfrac{x}{y}$ is

a) $\frac{2019}{4}$

b) $\frac{2019}{2}$

c) 2017

d) 2020

e) 4037

3. A drawer contains 10 socks and 5 of them are red. When three socks are drawn at random, what is the probability that all three of them are red?

a) $\frac{3}{4}$

b) $\frac{2}{13}$

c) $\frac{5}{37}$

d) $\frac{1}{2}$

e) $\frac{1}{12}$

4. If $x$ and $y$ are positive numbers that satisfy

$$\begin{cases} x^2 - y^2 = 24 \\ x^2 + y^2 = 26 \end{cases}$$

then $x + y$ is

a) 3

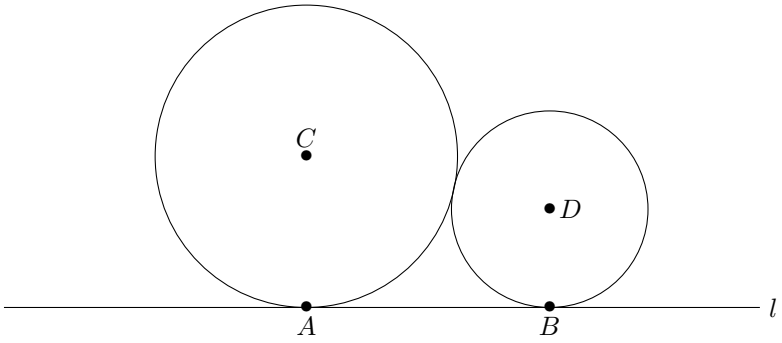
b) 4

c) 5

d) 6

e) none of these

5. The larger circle of radius 9 centered at $C$ is tangent to the smaller circle of radius 4 centered at $D$.




If the circles are tangent to the line $l$ at $A$ and $B$, then what is the area of the triangle $\Delta ABC$?

a) 36

b) 47

c) 54

d) 67

e) none of these

6. If $\theta$ is a solution of the equation

$$\cos(50^\circ) + \cos(70^\circ) = \cos\theta^\circ$$

then $\cos(9\theta^\circ)$ is

a) 0

b) $\frac{1}{2}$

c) $\frac{1+\sqrt{3}}{2}$

d) $\frac{\sqrt{3}}{2}$

e) 1

7. There are four boxes full of \$ 10, \$ 20 \$ 50, \$ 100 bills


You can choose your boxes and take 2,3,4, and 5 bills from the chosen boxes respectively. What is the largest amount of money you can get? (For example, if you take 2 bills from the \$100-box, 3 bills from the \$ 50-box, 4 bills from the \$ 20-box and 5 bills from §10-box, you would get \$480.)

a) \$ 480

b) \$ 560

c) \$ 780

d) \$ 850

e) \$ 920

8. If $x$ and $y$ are positive numbers satisfying

$$\log_3(x^2y^2) = 2\log_3 y + 6$$

then $x$ is

a) 1

b) 3

c) 9

d) 27

e) cannot be determined.

9. Suppose that $\triangle ABC$ is a triangle, that $M$ is the midpoint of $\overline{AC}$, and the segments $\overline{AM}$, $\overline{MC}$, $\overline{MB}$ and $\overline{AB}$ all have length 1. Find the area of $\triangle ABC$.

a) $\frac{1}{2}$

b) $\frac{\sqrt{2}}{2}$

c) $\frac{3}{2}$

d) $\frac{\sqrt{3}}{2}$

e) none of these.

10. If $\sin x = 2\cos x$, then what is $(\sin x)(\cos x)$?

a) $\frac{1}{2}$

b) $\frac{2}{3}$

c) $\frac{2}{5}$

d) $\frac{6}{7}$

e) $\frac{3}{11}$

11. If the sum of two of the roots of $x^3 - ax^2 + bx - c = 0$ is zero, then $ab - c$ is

a) $-2$

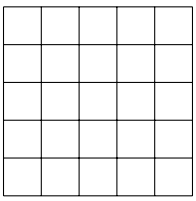
b) $-1$

c) 0

d) 1

e) 2

12. Suppose that the 25 squares below are filled with the integers 1 through 25 such a way that the sums of integers in each row and column are the same.



What is the common sum?

a) 65

b) 72

c) 83

d) 96

e) none of these.

13. The remainder when $6^{30} + 8^{30}$ is divisible by 49 is

a) 0

b) 2

c) 11

d) 12

e) 13

14. If a polynomial $p(x)$ of degree $\geq 1$ satisfies

$$p(p(x)) = 5p(x^3)$$

then the degree of $p(x)$ must be

a) 2

b) 3

c) 5

d) 7

e) none of these.

15. What is the remainder when $2^0 + 2^1 + 2^2 + \cdots + 2^{99}$ is divided by 9?

a) 0

b) 1

c) 3

d) 6

e) none of these.

16. What is the number of positive integer solutions $(x, y)$ of the equation

$$x^2 + y^2 = 3x + 3y + 4 - 2xy?$$

a) 0

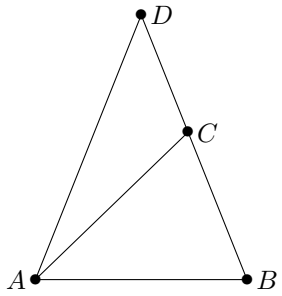
b) 1

c) 2

d) 3

e) none of these.

17. Consider the triangles in the following diagram:




Suppose that $\triangle ABC$ is isosceles with $AB = AC$ and $\triangle ABD$ is isosceles with $AD = BD$. Suppose that $AD = 9$ and $BC = 4$. Then $AB$ is

a) 6

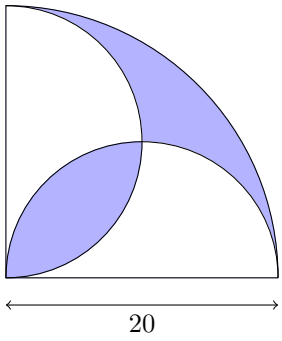
b) $6\sqrt{2}$

c) 8

d) $8\sqrt{2}$

e) 9.

18. Find the area of the shaded region formed by a large quarter circle of radius 20 and two smaller semicircles as shown below.




a) $100\pi - 200$

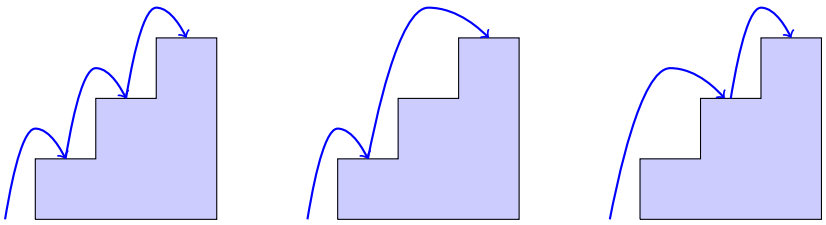
b) $200\pi - 100$

c) $200\pi + 100$

d) $150\pi + 100$

e) none of these.

19. In how many ways can you walk to a stairway with 7 steps if you can take one or two steps. (For example, you can walk up a stairway with 3 steps in three different ways: i) three 1 steps ii) 1 step and then 2 steps and iii) 2 steps and then 1 step.)



a) 15

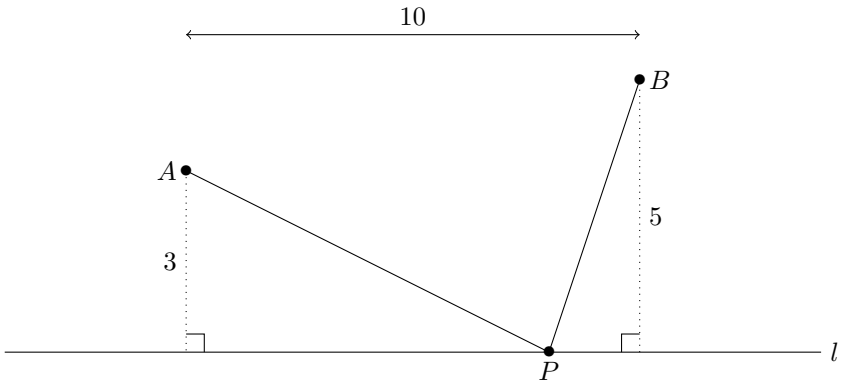
b) 18

c) 21

d) 27

e) 29

20. Two points $A$ and $B$ are vertically 3 and 5 ft away from the line $l$, respectively, and they are horizontally 10 ft apart.




When a point $P$ moves along the line $l$, what is the smallest value of $AP + PB$?

a) $\sqrt{97}$

b) $2\sqrt{41}$

c) $3\sqrt{43}$

d) $4\sqrt{45}$

e) $\sqrt{185}$.

21. The sum of the solutions to

$$2\Big(x + \frac{1}{x}\Big)^2 + 27 = 21\Big(x + \frac{1}{x}\Big)$$

is

a) $\frac{2}{3}$

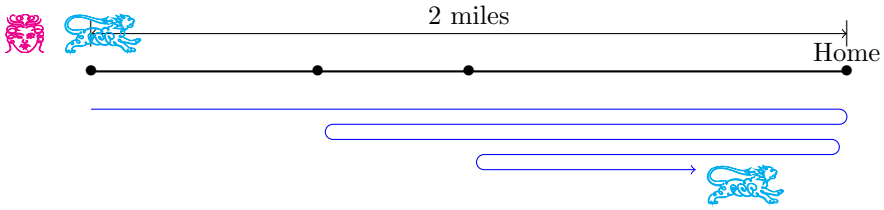
b) $-\frac{5}{7}$

c) $\frac{7}{6}$

d) $\frac{21}{2}$

e) $\frac{29}{5}$.

22. A student walks home from her school at a speed of $\frac{3}{2}$ miles per hour. She was greeted by her dog that runs 8 miles per hour when she was 2 miles away from home. The dog goes back home and then comes back to her and it keeps doing it until she arrives home.




What is the total distance the dog runs? (We assume that she and her dog keep the same paces $\frac{3}{2}$ miles per hour and 8 miles per hour, respectively.)

a) 10

b) $\frac{29}{3}$

c) $\frac{32}{3}$

d) $\frac{33}{2}$

e) $\frac{52}{5}$

23. If $f(x) = -x^2 + ax + b$ has the maximum value 2019 at $x = 3$, then $b$ is

a) 2010

b) 2015

c) 2018

d) 2020

e) 2024

24. Let $S_n$ be the sum of the first $n$ terms of the sequence $a_n$. If $S_n = 2^n$ for all $n \geq 1$, then what is $a_{2019}$?

a) $2^{10}$

b) $2^{20}$

c) $2^{2018}$

d) $2^{2019}$

e) none of these

25. Let $p$ and $q$ be prime numbers such that

$$p^2 - 2q^2 = 1$$

The number of such pairs $(p, q)$ is

a) 0

b) 1

c) 2

d) 11

e) infinitely many

26. For all values of $x$, the function $f(x)$ satisfies

$$f(x+1) + f(x-1) = f(x).$$

If $f(0) = 3$ and $f(1) = 5$ then $f(2018) + f(2019) + f(2020)$ is

a) $-3$

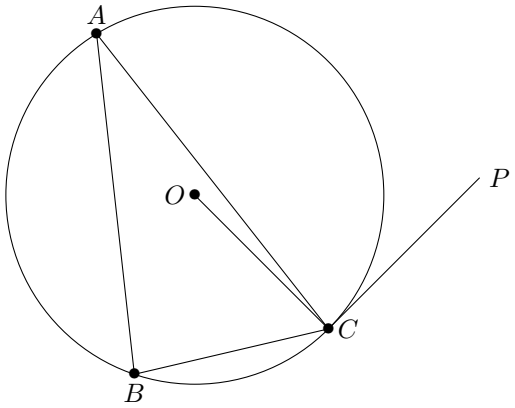
b) 0

c) 5

d) 8

e) none of these

27. Consider the following diagram of shapes in a circle:




The points $A$, $B$ and $C$ are on the circle, $O$ is the center of the circle, and the line $\overleftrightarrow{CP}$ is tangent to the circle. If $\angle ABC = 80°$, then $\angle ACP$ is

a) $60°$

b) $75°$

c) $90°$

d) $100°$

e) none of these.

28. A sequence $x_n$ is defined by

$$x_1 = 3 \qquad \text{and} \qquad x_{n+1} = \frac{x_n}{1 + nx_n} \qquad \text{for all } n \geq 1.$$

Then $x_{15}$ is

a) $\frac{3}{311}$

b) $\frac{3}{316}$

c) $\frac{3}{354}$

d) $\frac{3}{367}$

e) $\frac{3}{397}$

29. For integers $x$ and $y$ satisfying $xy = 3x + 2y + 23$, what is the largest possible value of $x - y$?

a) 23

b) 25

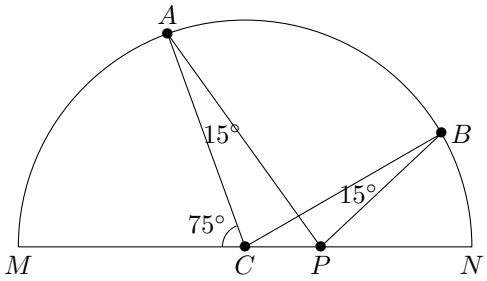
c) 27

d) 29

e) none of these

30. Two distinct points $A$ and $B$ are on a semicircle with diameter $MN$ and center $C$. The point $P$ is on $\overline{CN}$ and

$$\angle CAP = \angle CBP = 15^\circ.$$




If $\angle ACM = 75^\circ$ then $\angle BPN$ equals

a) $55^\circ$

b) $60^\circ$

c) $65^\circ$

d) $70^\circ$

e) $75^\circ$

(∗) Turn over for the tie breaker.

## Tie breakers

**Name:** **School Code:**

**After working on these problems, tear off this sheet and hand it in along with your answer form.**

a) Factor the following to the product of two polynomials on $a, b$.

$$a^3 + b^3 - ab(a + b).$$

b) Use a) or other means to prove that, for all nonnegative real numbers $a, b$,

$$a^3 + b^3 - ab(a + b) \geq 0.$$

c) Prove that, for all nonnegative real numbers $a, b, c$ and any positive integer $n$,

$$a^n(a - b)(a - c) + b^n(b - c)(b - a) + c^n(c - a)(c - b) \geq 0.$$