Minnesotata State University, Mankato
The 46th Annual High School Mathematics Contest
Problems and Solutions

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2. Problems and Solutions

1. If $x+y=7$ and $xy=6$ then $\frac{1}{x}+\frac{1}{y}$ is

a) $\frac{6}{7}$ *b)* $\frac{7}{6}$ *c)* $\frac{12}{7}$ *d)* $\frac{12}{14}$ *e)* $\frac{18}{7}$.

Answer-b) Observe that

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{7}{6}.$$

2. If $\frac{x-y}{x+y}=\frac{2018}{2019}$ then $\frac{x}{y}$ is

a) $\frac{2019}{4}$ *b)* $\frac{2019}{2}$ *c)* 2017 *d)* 2020 *e)* 4037.

Answer-e) Observe that

$$\frac{x-y}{x+y}=\frac{2018}{2019} \implies \frac{\frac{x}{y}-1}{\frac{x}{y}+1}=\frac{2018}{2019}$$

which implies

$$2018\cdot\frac{x}{y}+2018=2019\cdot\frac{x}{y}-2019 \implies \frac{x}{y}=4037.$$

3. A drawer contains 10 socks and 5 of them are red. When three socks are drawn at random, what is the probability that all three of them are red?

a) $\frac{3}{4}$ *b)* $\frac{2}{13}$ *c)* $\frac{5}{37}$ *d)* $\frac{1}{2}$ *e)* $\frac{1}{12}$.

Answer-e) The probability is

$$p=\frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}=\frac{1}{12}.$$

4. If x and y are positive numbers that satisfy

$$\begin{cases} x^2 - y^2 = 24 \\ x^2 + y^2 = 26 \end{cases}$$

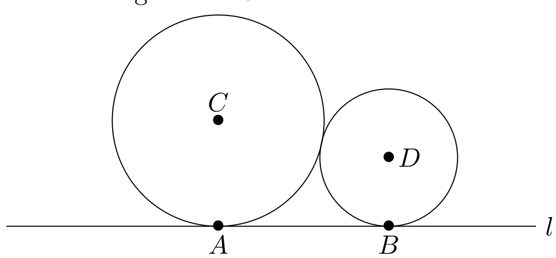
then $x + y$ is

a) 3 *b*) 4 *c*) 5 *d*) 6 *e*) none of these.

Answer-d) Observe that

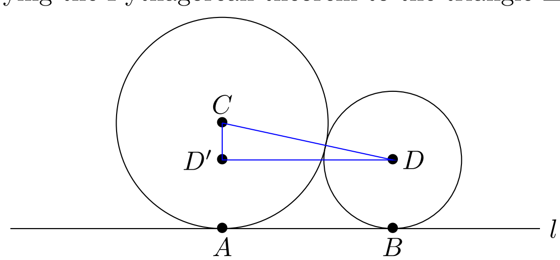
$$\begin{cases} x^2 - y^2 = 24 \\ x^2 + y^2 = 26 \end{cases} \implies \begin{cases} 2x^2 = 50 \implies x = 5 \\ -2y^2 = -2 \implies y = 1 \end{cases} \implies x + y = 6.$$

5. The larger circle of radius 9 centered at C is tangent to the smaller circle of radius 4 centered at D. If the circles are tangent to the line l at A and B, then what is the area of the triangle ΔABC?



a) 36 *b*) 47 *c*) 54 *d*) 67 *e*) none of these.

Answer-c) Appying the Pythagorean theorem to the triangle $\Delta CD'D$



we find that

$$AB = DD' = \sqrt{(9+4)^2 - (9-4)^2} = 12.$$

The area of the triangle is $\frac{1}{2} \cdot 12 \cdot 9 = 54$.

6. If θ is a solution of the equation

$$\cos(50^\circ) + \cos(70^\circ) = \cos\theta^\circ$$

then $\cos(9\theta^\circ)$ is

a) 0 *b*) $\frac{1}{2}$ *c*) $\frac{1+\sqrt{3}}{2}$ *d*) $\frac{\sqrt{3}}{2}$ *e*) 1.

Answer-a) Using the addition formulas

$$\begin{cases} \cos 70 = \cos(60+10) = \cos 60 \cos 10 - \sin 60 \sin 10 \\ \cos 50 = \cos(60-10) = \cos 60 \cos 10 + \sin 60 \sin 10 \end{cases}$$

we find that

$$\cos(50) + \cos(70) = 2\cos 60 \cos 10 = \cos 10 \implies \theta = \pm 10 \mod 360.$$

The answer is $\cos(9\theta) = \cos(\pm 90) = 0$.

7. There are four boxes full of \$ 10, \$ 20 \$ 50, \$ 100 bills

\$ 100 bills	\$ 50 bills	\$ 20 bills	\$ 10 bills

You can choose your boxes and take 2,3,4, and 5 bills from the chosen boxes respectively. What is the largest amount of money you can get? (For example, if you take 2 bills from the \$100-box, 3 bills from the \$ 50-box, 4 bills from the \$ 20-box and 5 bills from §10-box, you would get \$480.)

a) \$480 *b*) \$560 *c*) \$780 *d*) \$850 *e*) \$920.

Answer-c) By using the greedy algorithm, we find the largest sum is

$$5 \cdot 100 + 4 \cdot 50 + 3 \cdot 20 + 2 \cdot 10 = 780.$$

8. If x and y are positive numbers satisfying

$$\log_3(x^2y^2) = 2\log_3 y + 6$$

then x is

a) 1 *b*) 3 *c*) 9 *d*) 27 *e*) cannot be determined

Answer-d) [ANT:E: MMC 2017-15] Observe that

$$\log_3(x^2y^2) = 2\log_2 y + 2 \implies 2\log_3 x = 6 \implies x = 27.$$

9. Suppose that $\triangle ABC$ is a triangle, that M is the midpoint of $\overline{AC}$, and the segments $\overline{AM}$, $\overline{MC}$, $\overline{MB}$ and $\overline{AB}$ all have length 1. Find the area of $\triangle ABC$.

a) $\frac{1}{2}$ *b*) $\frac{\sqrt{2}}{2}$ *c*) $\frac{3}{2}$ *d*) $\frac{\sqrt{3}}{2}$ *e*) none of these.

Answer-d) There are a variety of ways to solve this, but they all revolve around the fact that the condition $AM = MC = BM$ forces the triangle to have a right angle at $\angle ABC$. This is a theorem that is sometimes seen in high school geometry courses. It can also be derived using inequality arguments. Here it is also possible to notice that $\triangle AMB$ is equilateral and $\triangle BMC$ is isosceles, and then calculate the angle values based on the angle sum for a triangle. Once you have established that $\triangle ABC$ is right, the Pythagorean theorem can be used show that $\overline{BC}$ has length $\sqrt{3}$, and so the area of the triangle must be $\sqrt{3} \cdot 1 \cdot (1/2)$.

10. If $\sin x = 2\cos x$, then what is $(\sin x)(\cos x)$?

a) $\frac{1}{2}$ *b*) $\frac{2}{3}$ *c*) $\frac{2}{5}$ *d*) $\frac{6}{7}$ *e*) $\frac{3}{11}$

Answer-c) Observe that

$$\sin x = 2\cos x \implies \tan x = 2 \implies \begin{cases} \sin x = \pm\frac{2}{\sqrt{5}} \\ \cos x = \pm\frac{1}{\sqrt{5}}. \end{cases}$$

with same parity on signs. $\sin x \cos x = \frac{2}{5}$.

11. If the sum of two of the roots of $x^3 - ax^2 + bx - c = 0$ is zero, then $ab - c$ is

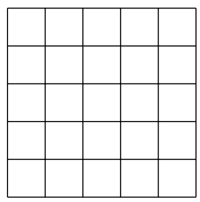
$a)$ -2 $b)$ -1 $c)$ 0 $d)$ 1 $e)$ 2.

Answer-c) Let α, β, γ be the roots of $x^3 - ax^2 + bx + c = 0$ and assume $\alpha + \beta = 0$. Then

$$\begin{cases} \alpha+\beta+\gamma = a \\ \alpha\beta+\beta\gamma+\gamma\alpha = b \\ \alpha\beta\gamma = c \end{cases} \implies \begin{cases} \gamma = a \\ \alpha\beta = b \\ \alpha\beta\gamma = c \end{cases} \implies ab = c.$$

The answer is (c).

12. Suppose that the 25 squares below are filled with the integers 1 through 25 such a way that the sums of integers in each row and column are the same. What is the common sum?



$a)$ 65 $b)$ 72 $c)$ 83 $d)$ 96 $e)$ none of these.

Answer-a) Let $x_1, x_2, .., x_{25}$ be integers in the magic square from left to right and from top to bottom. Let s be the common sum. Then

$$5s = (x_1+x_2+x_3+x_4+x_5)+\cdots+(x_{21}+x_{22}+x_{23}+x_{24}+x_{25}) = 1+2+\cdots+25 = 25\cdot 13$$

and so the sum is $s = 5 \cdot 13 = 65$.

13. The remainder when $6^{30} + 8^{30}$ is divisible by 49 is

$a)$ 0 $b)$ 2 $c)$ 11 $d)$ 12 $e)$ 13.

Answer-b) Using the binomial theorem to

$$6^{30} = (7-1)^{30} = 7^{30} - \binom{30}{1}7^{29} + \binom{30}{2}7^{28} + \cdots + \binom{30}{28}7^2 - \binom{30}{29}7 + 1$$

and

$$8^{30} = (7+1)^{30} = 7^{30} + \binom{30}{1}7^{29} + \binom{30}{2}7^{28} + \cdots + \binom{30}{28}7^2 + \binom{30}{29}7 + 1$$

we find that

$$6^{30} + 8^{30} = \text{ (a multiple of 49)} + 2.$$

The answer is 2.

14. If a polynomial $p(x)$ of degree ≥ 1 satisfies

$$p(p(x)) = 5p(x^3)$$

then the degree of $p(x)$ must be

$a)$ 2 $b)$ 3 $c)$ 5 $d)$ 7 $e)$ none of these.

Answer-b) Let n be the degree of $p(x)$. Then

$$n^2 = 3n \implies n = 3.$$

15. What is the remainder when $2^0 + 2^1 + 2^2 + \cdots + 2^{99}$ is divided by 9.

$a)$ 0 $b)$ 1 $c)$ 3 $d)$ 6 $e)$ none of these.

Answer-d) Note that $2^0 + 2^1 + 2^2 + \cdots + 2^{99} = 2^{100} - 1$ and $2^6 \equiv 1 \mod 9$. The sequence $2^n - 1 \mod 9$ is periodic with period 6 and $100 = 6 \cdot 16 + 4$, $2^{100} - 1 \equiv 2^4 - 1 = 6 \mod 9$. Elementary pattern search would work as well.

16. What is the number of positive integer solutions (x, y) of the equation

$$x^2 + y^2 = 3x + 3y + 4 - 2xy?$$

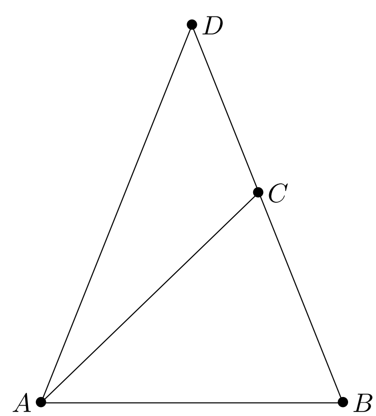
$a)$ 0 $b)$ 1 $c)$ 2 $d)$ 3 $e)$ none of these.

Answer-a) Observe that

$$x^2+y^2 = 3x+3y+4-2xy \implies (x+y)^2-3(x+y)-4 = 0 \implies (x+y+1)(x+y-4) = 0$$

There are 3 positive integer solutions (x, y) for $x + y = 4$.

17. Consider the triangles in the following diagram:



Suppose that $\triangle ABC$ is isosceles with $AB = AC$ and $\triangle ABD$ is isosceles with $AD = BD$. Suppose that $AD = 9$ and $BC = 4$. Then AB is

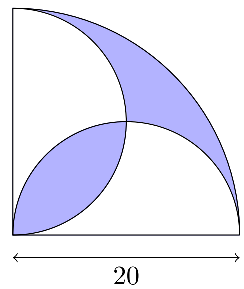
$a)$ 6 $b)$ $6\sqrt{2}$ $c)$ 8 $d)$ $8\sqrt{2}$ $e)$ 9.

Answer-a) Using the isosceles triangle theorem, the angles $\angle ABC$, $\angle ACB$ and $\angle DAB$ all must be congruent. Therefore we have the similarity $\triangle ABC \sim \triangle DAB$. This gives us the equation:

$$\frac{AB}{AD} = \frac{BC}{AB} \implies \frac{AB}{9} = \frac{4}{AB} \implies AB^2 = 36$$

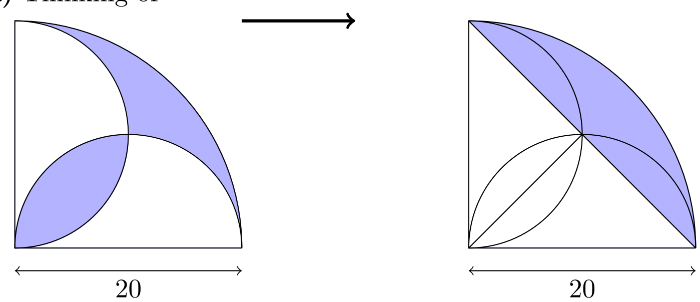
Since AB must be positive, this equation has the unique solution that $AB = 6$.

18. Find the area of the shaded region formed by a large quarter circle of radius 10 and two smaller semicircles as shown below.



$a)$ $100\pi - 200$ $b)$ $200\pi - 100$ $c)$ $200\pi + 100$ $d)$ $150\pi + 100$ $e)$ none of these.

Answer-a) Thinking of



we find that the area is $\frac{1}{4} \cdot \pi(20)^2 - \frac{1}{2} \cdot 20 \cdot 20 = 100\pi - 200$.

19. In how many ways can you walk to a stairway with 7 steps if you can take one or two steps. (For example, you can walk up a stairway with 3 steps in three different ways: i) three 1 steps ii) 1 step and then 2 steps and iii) 2 steps and then 1 step.)

$a)$ 15 $b)$ 18 $c)$ 21 $d)$ 27 $e)$ 29.

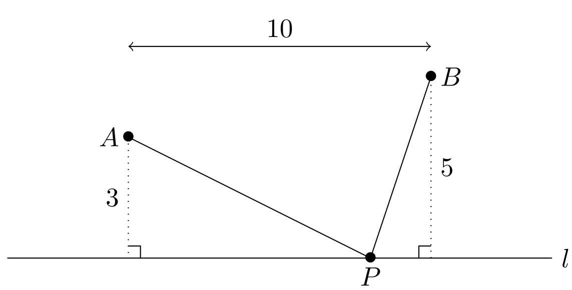
Answer-c) Let f_n be the number of ways to walk to a stairway with n steps. Then f_n satisfies

$$f_n = f_{n-1} + f_{n-2}$$

that is, f_n is a Fibonacci sequence. Since $f_1 = 1$ and $f_2 = 2$, we find that

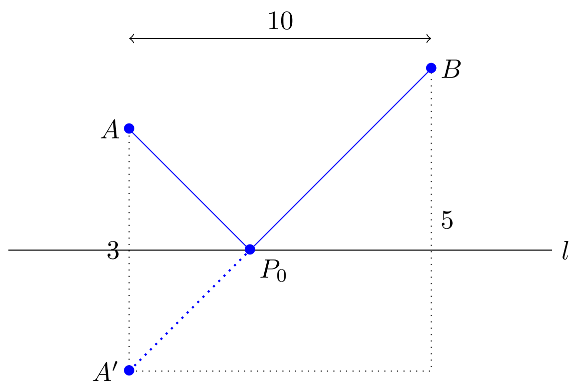
$$f_3 = 3 \implies f_4 = 5 \implies f_5 = 8 \implies f_6 = 13 \implies f_7 = 21.$$

20. Two points A and B are vertically 3 and 5 ft away from the line l, respectively, and they are horizontally 10 ft apart. When a point P moves along the line l, what is the smallest value of $AP + PB$.



a) $\sqrt{97}$ *b)* $2\sqrt{41}$ *c)* $3\sqrt{43}$ *d)* $4\sqrt{45}$ *e)* $\sqrt{185}$.

Answer-b) [G:N: None] Let A', B' be the reflections of A, B with respect to l and let P_0 be the point at which $A'B$ meets l. The sum $AP + PB$ minimizes at $P = P_0$.



Applying the Pythagorean theorem to the right triangle above, we find that

$$AP_0 + P_0B = A'B = \sqrt{10^2 + 8^2} = \sqrt{164} = 2\sqrt{41}.$$

21. The sum of the solutions to

$$2\left(x + \frac{1}{x}\right)^2 + 27 = 21\left(x + \frac{1}{x}\right)$$

a) $\frac{2}{3}$ *b)* $-\frac{5}{7}$ *c)* $\frac{7}{6}$ *d)* $\frac{21}{2}$ *e)* $\frac{29}{5}$.

Answer-d) Let $t = x + \frac{1}{x}$. Then

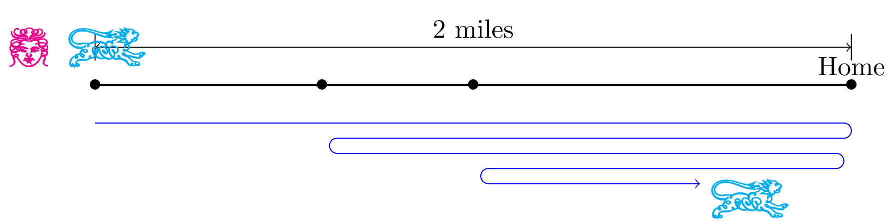
$$2t^2 + 27 = 2t \implies t = \frac{3}{2}, 9 \implies \begin{cases} x + \frac{1}{x} = \frac{3}{2} \implies x^2 - \frac{3}{2}x + 1 = 0 \\ x + \frac{1}{x} = 9 \implies x^2 - 9x + 1 = 0 \end{cases}$$

The sum is $\frac{3}{2} + 9 = \frac{21}{2}$.

Remark) If α, β are roots of a quadractic equation $ax^2 + bx + c = 0$ then

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

22. A student walks home from her school at a speed of $\frac{3}{2}$ miles per hour. She was greeted by her dog that runs 8 miles per hour when she was 2 miles away from home. The dog goes back home and then comes back to her and it keeps doing it until she arrives home.

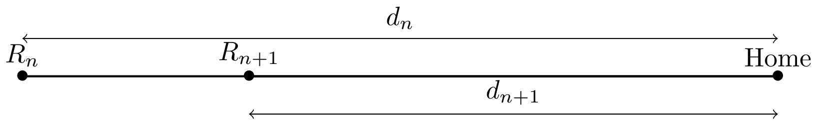


What is the total distance the dog runs? (We assume that she and her dog keep the same paces $\frac{3}{2}$ miles per hour and 8 miles per hour, respectively.)

a) 10 *b*) $\dfrac{29}{3}$ *c*) $\dfrac{32}{3}$ *d*) $\dfrac{33}{2}$ *e*) $\dfrac{52}{5}$

Answer-c) Solution I) The total distance she travels is 2 miles, and the ratio of the speed of her to that of the dog is $3:16$. The total distance of the dog's travel is $\frac{16}{3}\cdot 2 = \frac{32}{3}$.

Solution II) For each rendevous point R_n for $n \geq 1$, let t_n be the time it takes to go from R_n to R_{n+1}.



Then

$$\frac{3}{2}t_n = d_n - d_{n+1} \qquad \text{and} \qquad 8t_n = d_n + d_{n+1}$$

Eliminating t_n, we have

$$d_{n+1} = \frac{13}{19}d_n \implies (d_{n+1}+d_n) = \frac{13}{19}(d_n + d_{n-1})$$

The dog  runs

$$\sum_{n=1}^{\infty}(d_{n+1}+d_n) = \left[1+\frac{13}{19}+\left(\frac{13}{19}\right)^2+\cdots+\left(\frac{13}{19}\right)^n+\cdots\right](d_2+d_1) = \frac{1}{1-\frac{13}{19}}\cdot\frac{64}{19} = \frac{32}{3}.$$

23. If $f(x) = -x^2 + ax + b$ has the maximum value 2019 at $x = 3$, then b is

a) 2010 *b*) 2015 *c*) 2018 *d*) 2020 *e*) 2024

Answer-a) Completing the square

$$f(x) = -x^2 + ax + b = -\left(x - \frac{a}{2}\right)^2 + b + \frac{a^2}{4}$$

we find that $f(x)$ has the maximum value $b + \frac{a^2}{4}$ at $x = \frac{a}{2}$. The answer is $a = 6$ and $b = 2010$.

24. Let S_n be the sum of the first n terms of the sequence a_n. If $S_n = 2^n$ for all $n \geq 1$, then what is a_{2019}?

a) 2^{10} *b*) 2^{20} *c*) 2^{2018} *d*) 2^{2019} *e*) none of these.

Answer-c) Using the general formula $a_n = S_n - S_{n-1}$ for $n \geq 2$, we find that

$$a_{2019} = 2^{2019} - 2^{2018} = 2^{2018}.$$

25. Let p and q are prime numbers such that

$$p^2 - 2q^2 = 1$$

The number of such pairs (p, q) is

a) 0 *b*) 1 *c*) 2 *d*) 11 *e*) infinitely many.

Answer-b) It is clear that $p > q$ and that p must be odd because $p^2 = 1 + 2q^2$. This, in turn, implies q must be even because

$$2q^2 = p^2 - 1 = (p+1)(p-1) \implies 2q^2 = 4m \implies q^2 = 2m \qquad \text{for some integer } m.$$

The only solutions is $(p, q) = (3, 2)$.

26. For all values of x, the function $f(x)$ satisfies

$$f(x+1) + f(x-1) = f(x).$$

If $f(0) = 3$ and $f(1) = 5$ then $f(2018) + f(2019) + f(2020)$ is

a) -3 *b*) 0 *c*) 5 *d*) 8 *e*) none of these.

Answer-e) Observe that

$$f(x+1) = f(x) - f(x-1) \implies \begin{cases} f(x+2) = f(x+1) - f(x) \\ f(x+1) = f(x) - f(x-1) \end{cases}$$

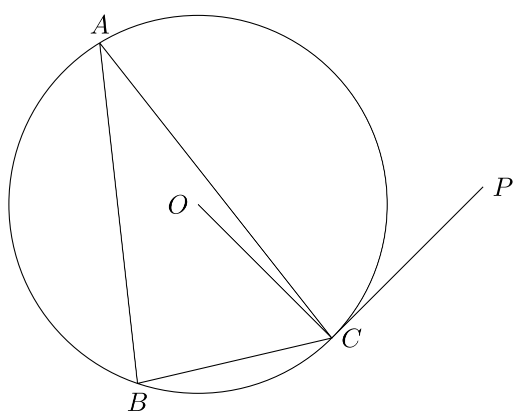
which implies

$$\implies f(x+2) = -f(x-1) \implies f(x+3) = -f(x) \implies f(x+6) = f(x).$$

Note that $f(-1) = f(0) - f(1) = -2$ and so $f(2) = -2$.

$$f(2018) + f(2019) + f(2020) = f(2) + f(3) + f(4) = 2 + (-3) + (-5) = -6$$

27. Consider the following diagram of shapes in a circle:



The points A, B and C are on the circle, O is the center of the circle, and the line $\overleftrightarrow{CP}$ is tangent to the circle. Furthermore $\angle ABC$ has measure of 80 degrees. What is the measures of $\angle ACP$?

a) 60° *b*) 75° *c*) 90° *d*) 100° *e*) None of these.

Solution-e) The measure is 80 degrees, since $\angle ABC$ and $\angle ACP$ are congruent. Sometimes this result is proven as an extension of the inscribed angle theorem for circles. (I.e. if we choose some point X between B and C on the circle then $\angle ABC$ and $\angle AXC$ would be congruent. The tangent angle is sort of a limit case of that). If students are familiar with that result, then the answer is immediate. If they are not familiar with that result then they can compare angles in the triangles by using the fact that $\triangle AOC$, $\triangle AOB$ and $\triangle BOC$ are isosceles and $\angle OCP$ must be right (since $\overleftrightarrow{CP}$ is tangent to the circle).

28. A sequence x_n is defined by

$$x_1 = 3 \qquad \text{and} \qquad x_{n+1} = \frac{x_n}{1 + nx_n} \qquad \text{for all } n \geq 1.$$

Then x_{15} is

a) $\frac{3}{311}$ *b)* $\frac{3}{316}$ *c)* $\frac{3}{354}$ *d)* $\frac{3}{367}$ *e)* $\frac{3}{397}$.

Answer-b) Observe that

$$x_{n+1} = \frac{x_n}{1 + nx_n} \implies \frac{1}{x_{n+1}} = \frac{1}{x_n} + n \implies \frac{1}{x_{n+1}} - \frac{1}{x_n} = n.$$

The alternating sequence method

$$\sum_{n=1}^{14} \left(\frac{1}{x_{n+1}} - \frac{1}{x_n} \right) = \sum_{n=1}^{14} n$$

gives

$$\frac{1}{x_{15}} - \frac{1}{x_1} = 1 + 2 + \cdots + 14 \implies \frac{1}{x_{15}} = \frac{316}{3}.$$

29. For integers x and y satisfying $xy = 3x + 2y + 23$, what is the largest possible value of $x - y$?

a) 23 *b)* 25 *c)* 27 *d)* 29 *e)* none of these.

Answer-c) Observe that

$$xy = 3x + 2y + 23 \implies (x-2)(y-3) = 29$$

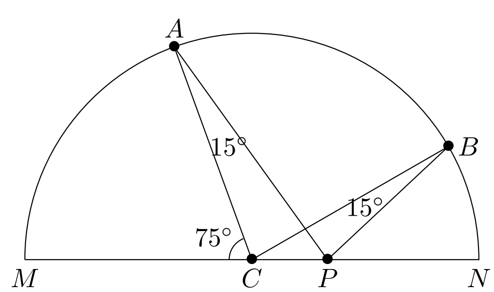
and so

$$\begin{cases} x-2 = 1, y-3 = 29 & \implies x = 3, y = 32 \\ x-2 = 29, y-3 = 1 & \implies x = 31, y = 4 \\ x-2 = -1, y-3 = -29 & \implies x = 1, y = -26 \\ x-2 = -29, y-3 = -1 & \implies x = 27, y = 2 \end{cases}$$

and so the largest $x - y$ is 27.

30. Two distinct points A and B are on a semicircle with diameter MN and center C. The point P is on $\overline{CN}$ and

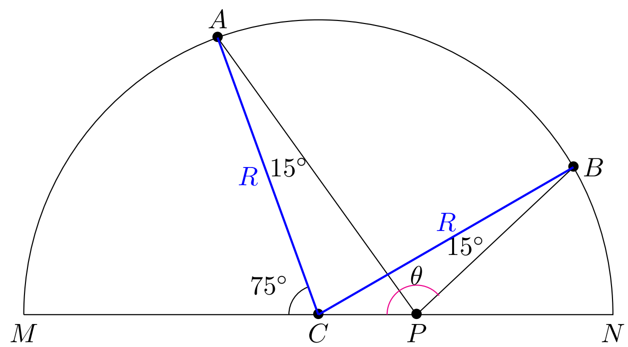
$$\angle CAP = \angle CBP = 15°.$$



If $\angle ACM = 75°$ then $\angle BPN$ equals

a) 55° *b*) 60° *c*) 65° *d*) 70° *e*) 75°.

Answer-b) Consider the figure below



Observe that

$$\angle CPA = 75 - 15 = 60.$$

Let R be the radius of the halfcircle and let $\theta = \angle CPB$. Applying the law of sines to ΔACP and to ΔCPB, we find that

$$\frac{\sin 60}{R} = \frac{\sin 15}{CP} \quad \text{and} \quad \frac{\sin\theta}{R} = \frac{\sin 15}{CP}$$

which implies

$$\sin\theta = \sin 60 \implies \theta = 180 - 60 = 120 \implies \angle BPN = 60.$$

Tie breakers. Answer the questions.

a) Factor the following to the product of two polynomials on a, b.

$$a^3 + b^3 - ab(a+b).$$

Solution) Using

$$a^3 + b^3 = (a+b)(a^2 + ab + b^2)$$

we factor

$$a^3 + b^3 - ab(a+b) = (a+b)(a^2+ab+b^2) - ab(a+b) = (a+b)(a^2+b^2).$$

b) Use a) or other means to prove that, for all nonnegative real numbers a, b,

$$a^3 + b^3 - ab(a+b) \geq 0.$$

solution) It is clear from the factoring of b) that

$$a^3 + b^3 - ab(a+b) = (a+b)(a^2+b^2) \geq 0.$$

c) Prove that, for all nonnegative real numbers a, b, c and any positive integer n,

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geq 0.$$

Solution) Since the inequality is symmetric on a, b, c, we may assume that $a \geq b \geq c$. Observe that

$$\begin{aligned} LHS &= a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \\ &= (a-b)[a^n(a-c) - b^n(b-c)] + c^n(c-a)(c-b) \geq 0 \end{aligned}$$

because

$$a^n(a-c) \geq b^n(a-c) \geq b^n(b-c) \implies a^n(a-c) - b^n(b-c) \geq 0.$$