Minnesotata State University, Mankato
The 47th Annual High School Mathematics Contest
April 19, 2021

Instruction for the Contest

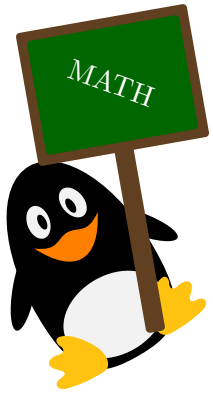
1. You have 90 minutes to work for 30 questions and 2 tie breaker problems.

2. The last two Tie Breaker problems TB I and TB II will be used only to break any possible ties that arise on the test.

3. This is a multiple choice test and there is no penality for a wrong answer.

4. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

5. Submit your answer sheet or the booklet with answers to your math coach or teacher at the end of the contest.



The Answer Sheet

Question	1	2	3	4	5	6	7	8	9	10
Answer										
Question	11	12	13	14	15	16	17	18	19	20
Answer										
Question	21	22	23	24	25	26	27	28	29	30
Answer										

Question	**TB I**	**TB II**
Answer		

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1. Alice sells an item at \$10 less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price in dollars is

a) \$20

b) \$30

c) \$50

d) \$70

e) \$100

2. For the solution x, y and z of the folowing system of equations

$$x + y - z = 0$$
$$y + z - x = 4$$
$$z + x - y = 2$$

find the product xyz.

a) -1

b) 0

c) 1

d) 2

e) 6

3. Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

a) Andy

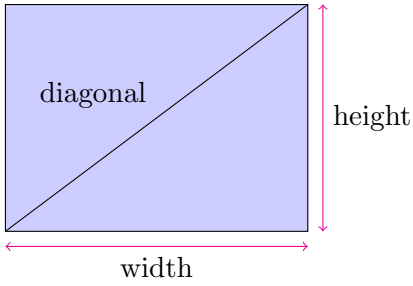
b) Beth

c) Carlos

d) Andy and Carlos tie for first.

e) All three tie.

4. Many television screens are rectangles that are measured by the length of their diagonals. The ratio of the width to the height in an old standard television screen is 4 : 3.



(width):(height)=4:3

The width of an old 27-inch television screen is closest, in inches, to which of the following?

a) 20

b) 20.5

c) 21

d) 21.5

e) 22

5. When 2^{2021} is divided by 7, the remainder is

a) 1

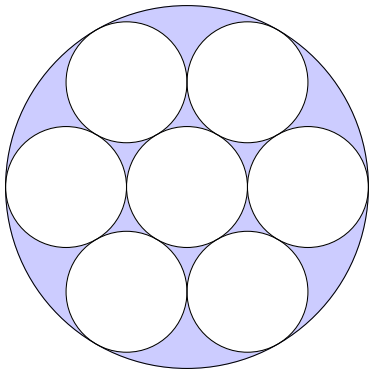
b) 2

c) 3

d) 4

e) 5

6. Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors.



Find the area of the shaded region.

a) π

b) 1.5π

c) 2π

d) 3π

e) 3.5π

7. If x, y are real numbers satisfying $x^2 + y^2 = 2$, what is the maximum value of $2x + 2y$?

a) 1

b) 2

c) 3

d) 4

e) 5

8. Let f be a function with the following properties:
i) $f(1) = 1$ and
ii) $f(2n) = n \cdot f(n)$ for any positive integer n.
What is the value of $f(2^{10})$?

a) 1

b) 2^{10}

c) 2^{25}

d) 2^{30}

e) 2^{45}

9. How many integers between 1 and 2021 (inclusive) share a prime factor with 10?

a) 404

b) 607

c) 809

d) 1212

e) 1414

10. Fairies on the planet Twinkle Star have an alphabet with only the three letters A, B, C. They want to make licence plates for flying cars that use four letters $\boxed{}\boxed{}\boxed{}\boxed{}$. (For example, $\boxed{A}\boxed{B}\boxed{C}\boxed{C}$ and $\boxed{C}\boxed{C}\boxed{A}\boxed{B}$ are possible plates and they are considered different.) How many plates can they make?

a) 3

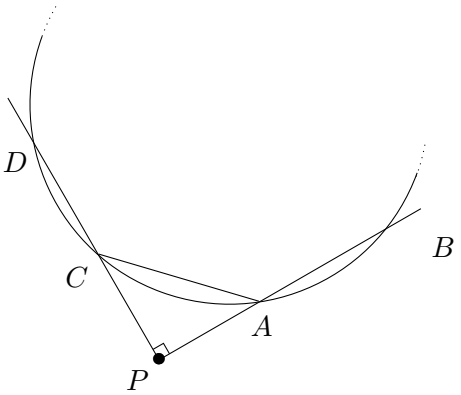
b) 9

c) 15

d) 27

e) 81

11. Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines which intersect a circle as shown in the figure below



Furthermore, suppose that the two lines intersect at a right angle at a point P outside the circle, $PA = 4$, $AB = 6$ and $CD = 3$. Find the length AC.

a) 6

b) $\sqrt{41}$

c) $\sqrt{47}$

d) 7

e) $\sqrt{53}$

12. What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

a) 20

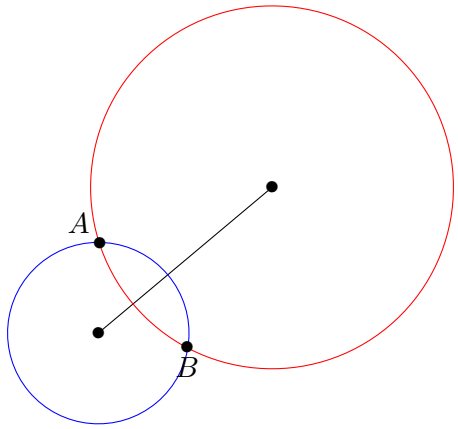
b) 25

c) 10

d) 30

e) 45

13. Suppose that two cicles of radii 7 and 13 intersect at the points A and B.



If $AB = 10$, what is the distance between the center of the two circles?

a) $2(\sqrt{6} + 6)$

b) $3(\sqrt{7} + 1)$

c) 16

d) 20

e) $4(\sqrt{11} + 1)$

14. The sum of 18 consecutive positive integers is a perfect square. (A perfect square is an integer square such as $4 = 2^2$ or $25 = 5^2$.) The smallest possible value of this sum is

a) 169

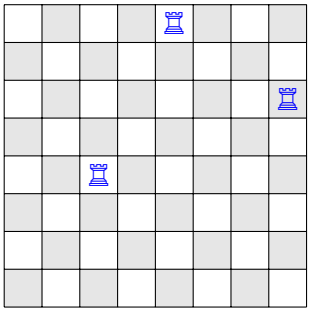
b) 225

c) 289

d) 361

e) 441

15. Chess is played on an 8×8 board of squares.



How many ways are there to put 3 identical rooks on the chessboard so that no two rooks are in the same row or column?

a) 56

b) 336

c) 6720

d) 18816

e) 112896

16. How many prime numbers are there between $2021! + 2$ and $2021! + 2021$? (Here, 2021! is the product of all natural numbers between 1 and 2021, i.e.,

$$2021! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2017 \cdot 2018 \cdot 2019 \cdot 2020 \cdot 2021 \quad).$$

a) 0

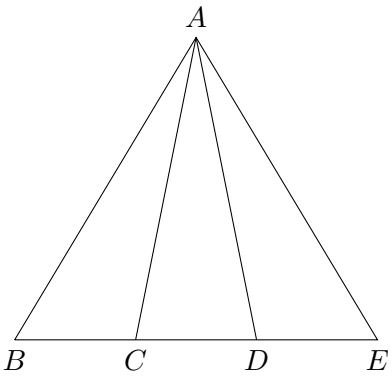
b) 1

c) 5

d) 19

e) 35

17. Suppose that we have points arranged as in the following figure:



If $AB = AE$ and $BC = CD = DE$, which of the following is true?

a) $\angle BAC = \angle EAD = \angle CAD$

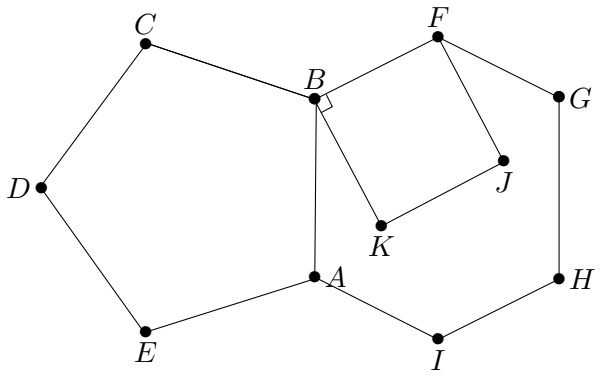
b) $\angle BAC = \angle EAD < \angle CAD$

c) $\angle BAC = \angle EAD > \angle CAD$

d) $\angle BAC > \angle EAD > \angle CAD$

e) $\angle EAD > \angle BAC > \angle CAD$

18. A regular pentagon $ABCDE$ and a regular hexgon $ABFGHI$ share a side AB and a square $BFJK$ is placed in the interior of the hexagon. What is $\angle CBK$?



a) 120°

b) 134°

c) 136°

d) 138°

e) 140°

19. If θ is a solution of the equation

$$\sin(195^\circ) + \sin(105^\circ) = \sin\theta$$

then $\cos(2\theta)$ is

a) -1

b) $-\frac{1}{2}$

c) $-\frac{\sqrt{3}}{2}$

d) 0

e) $-\frac{\sqrt{6}}{2}$

20. How many real numbers x satisfy

$$\sin x = \frac{x}{6\pi} \quad ?$$

a) 9

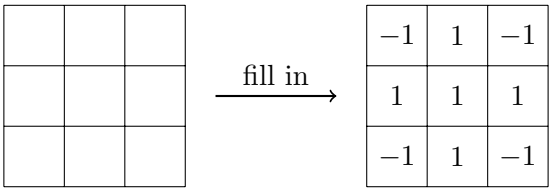
b) 11

c) 13

d) 15

e) 16

21. In each cell of the following 3×3 table, fill in 1 or -1, such that the product of numbers in each row or column is equal to 1. For example,



How many ways are there to fill in this table?

a) 12

b) 16

c) 24

d) 36

e) 18

22. The remainder of $3^{21} + 5^{21}$ divided by 64 is

a) 4

b) 20

c) 12

d) 10

e) 40

23. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walters entered?

a) 71

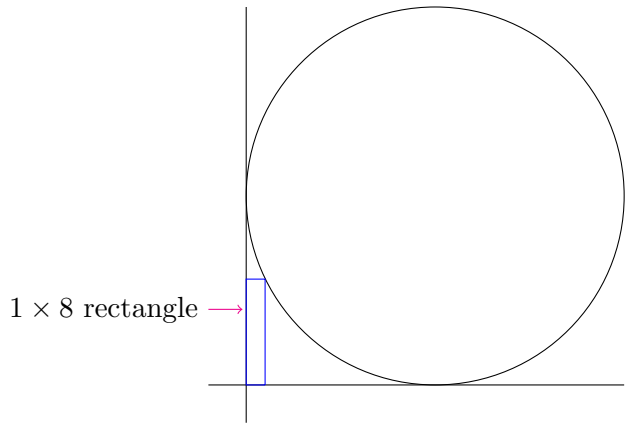
b) 76

c) 80

d) 82

e) 91

24. Suppose that a circle is tangent to two perpendicular lines so that a 1×8 rectangle can be placed between the perpendicular lines and the circle as in the following figure



What is the radius of the circle?

a) 7

b) 9

c) 12

d) 13

e) 16

25. Suppose $x \neq 1$. The product of real numbers x satisfying

$$\log_2 x - 1 - \log_x 4 = 0$$

is

a) 8

b) 1

c) 2

d) 6

e) 3

26. Let $f(x)$ denote the sum of the digits of the positive integer x. For example, $f(8) = 8$ and $f(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $f(f(x)) = 3$?

a) 3

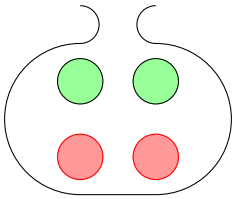
b) 4

c) 6

d) 9

e) 10

27. A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out.



What is the probability that all beads in the bag are red after three such replacements?

a) 1/8

b) 5/32

c) 9/32

d) 3/8

e) 7/16

28. Let $f(x)$ be a function not defined for $x = 0$ but

$$f(x) + 2f\Big(\frac{1}{x}\Big) = 3x \qquad \text{for all nonzero } x.$$

Find the solutions of the equation

$$f(x) = f(-x).$$

a) $x = \pm 1$

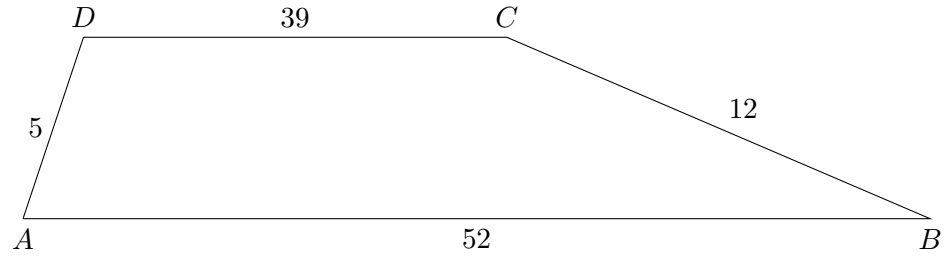
b) $x = \pm\sqrt{2}$

c) $x = \pm\sqrt{3}$

d) $x = \pm\sqrt{5}$

e) $x = \pm\sqrt{7}$

29. In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$.



The area of $ABCD$ is

a) 182

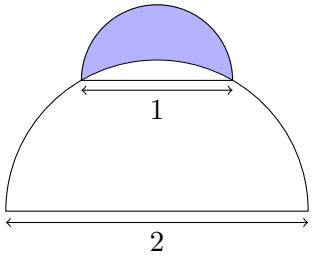
b) 195

c) 210

d) 234

e) 260

30. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2 as shown below.



The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune.

a) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

b) $\frac{\sqrt{3}}{4} - \frac{\pi}{12}$

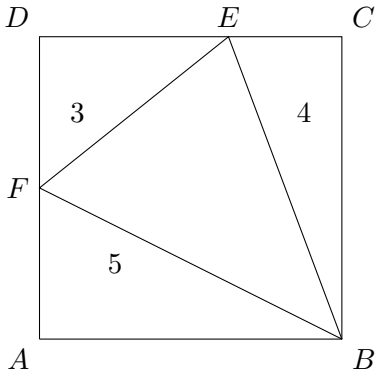
c) $\frac{\sqrt{3}}{4} - \frac{\pi}{24}$

d) $\frac{\sqrt{3}}{4} + \frac{\pi}{24}$

e) $\frac{\sqrt{3}}{4} + \frac{\pi}{12}$

Look at the next page for the Tie Breaker questions.

Tie Breaker (TB) problems: In a square $ABCD$, the areas of $\triangle EDF$, $\triangle BCE$ and $\triangle FAB$ are 3, 4, and 5 respectively.



TB I) What is the area of $\triangle EFB$?

a) 6.5

b) 6

c) 5.5

d) 7

e) 8

TB II) What is the value of $\cos(\angle FEB)$?

a) $\frac{12}{\sqrt{1769}}$

b) 0

c) $\frac{13}{\sqrt{1769}}$

d) $\frac{14}{\sqrt{1769}}$

e) $\frac{12}{\sqrt{1767}}$