Minnesotata State University, Mankato
The 47th Annual High School Mathematics Contest
Answers and Solutions

Instruction for the Contest

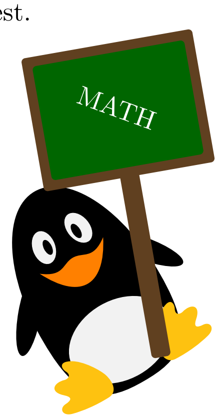
1. You have 90 minutes to work for 30 questions and 2 tie breaker problems.

2. The last two Tie Breaker problems TB I and TB II will be used only to break any possible ties that arise on the test.

3. This is a multiple choice test and there is no penality for a wrong answer.

4. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

5. Submit your answer sheet or the booklet with answers to your math coach or teacher at the end of the contest.



The Answer Key

Question	1	2	3	4	5	6	7	8	9	10
Answer	b	e	b	d	d	c	d	e	d	e
Question	11	12	13	14	15	16	17	18	19	20
Answer	b	a	a	b	d	a	b	d	d	b
Question	21	22	23	24	25	26	27	28	29	30
Answer	b	e	c	d	c	e	c	b	c	c

Question	**TB I**	**TB II**
Answer	e	c

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April 19, 2021

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1. Alice sells an item at \$10 less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price in dollars is

a) \$20

b) \$30

c) \$50

d) \$70

e) \$100

Answer: b)

Solution) If x is the list price, then $10\% \cdot (x - 10) = 20\% \cdot (x - 20)$ or equivalently

$$\frac{10}{100} \cdot (x - 10) = \frac{20}{100} \cdot (x - 20) \implies x - 10 = 2(x - 20) \implies x = 30.$$

2. For the solution x, y and z of the folowing system of equations

$$\begin{aligned} x + y - z &= 0 \\ y + z - x &= 4 \\ z + x - y &= 2 \end{aligned}$$

find the product xyz.

a) -1

b) 0

c) 1

d) 2

e) 6

Answer: e)

Solution I) Adding three equations, we find that

$$x+y+z=6 \implies \begin{cases} (x+y)-z=0 \implies (6-z)-z=0 \implies z=3 \\ (y+z)-x=4 \implies (6-x)-x=4 \implies x=1 \\ (z+x)-y=2 \implies (6-y)-y=2 \implies y=2 \end{cases}$$

and so the product is $xyz = 1 \cdot 2 \cdot 3$.

Solution II) Since the first equation is the same as $z = x + y$, we may work with the second and third equations as follow

$$\begin{cases} z = x+y \\ y+z-x=0 \\ z+x-y=2 \end{cases} \implies \begin{cases} z = x+y \\ y+(x+y)-x=4 & \implies y=2 \\ (x+y)+x-y=2 & \implies x=1 \end{cases}$$

and obtain $z = 1+2 = 3$. The product is $xyz = 1 \cdot 2 \cdot 3$. Of course, there are many other ways (including the Gaussian elimination method) to solve the system.

3. Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

a) Andy

b) Beth

c) Carlos

d) Andy and Carlos tie for first.

e) All three tie.

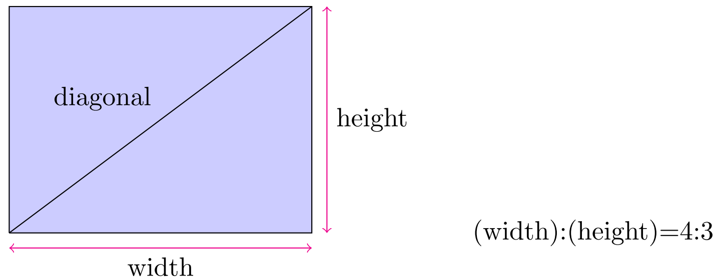
Answer: b)

Solution) We say Andy's lawn has an area of A. Beth's lawn thus has an area of $\frac{A}{2}$, and Carlos' lawn has an area of $\frac{A}{3}$. We say Andy's lawn mower cuts at a speed of v in the unit $\frac{\text{area}}{\text{time}}$. Carlos' cuts at a speed of $\frac{v}{3}$, and Beth's cuts at a speed of $\frac{2v}{3}$.

Each person's lawn is cut at a time of $\frac{\text{area}}{\text{speed}}$. Andy's lawn is cut in $\frac{A}{v}$ time, Beth's lawn is cut in $\frac{\frac{A}{2}}{\frac{2v}{3}} = \frac{3}{4} \cdot \frac{A}{v}$ time, and Carlos' lawn is cut in $\frac{\frac{A}{3}}{\frac{v}{3}} = \frac{A}{v}$. Therefore, Beth finishes first.

4. Many television screens are rectangles that are measured by the length of their

diagonals. The ratio of the width to the height in an old standard television screen is $4:3$.



The width of an old 27-inch television screen is closest, in inches, to which of the following?

a) 20

b) 20.5

c) 21

d) 21.5

e) 22

Answer: d)

Solution) The triangle with hypotenuse as the diagonal is a right triangle and so, by Pythagorean's theorem, we have the ratios

$$\text{(height): (length): (diagonal)} = (3:4:5).$$

The horizontal length of the right triangle with hypotenuse 27 is

$$\frac{4}{5} \times 27 = 21.6 \approx 21.5.$$

5. When 2^{2021} is divided by 7, the remainder is

a) 1

b) 2

c) 3

d) 4

e) 5

Answer: d)

Solution) Observe that $2^3 \equiv 1 \mod 7$. By division of 2021 by 3

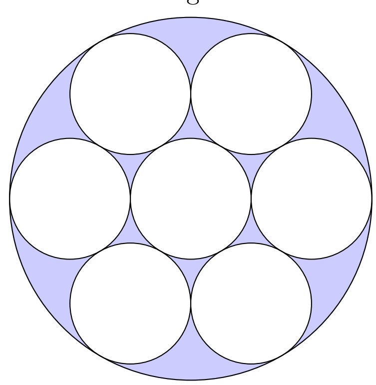
$$2021 = 3 \cdot 673 + 2$$

we find that

$$\begin{aligned} 2^{2021} &\equiv 2^{3\cdot 673+2} \\ &\equiv (2^3)^{673} \cdot 2^2 \\ &\equiv 1^{673} \cdot 2^2 \\ &\equiv 2^2 \\ &\equiv 4 \mod 7. \end{aligned}$$

Remark) The problem is simplified (again because of the length of the exam) but uses similar mathematical ideas. I wanted to avoid lengthy computation or Fermat's little theorem. Is the original problem easy? medium?

6. Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors.



Find the area of the shaded region.

a) π

b) 1.5π

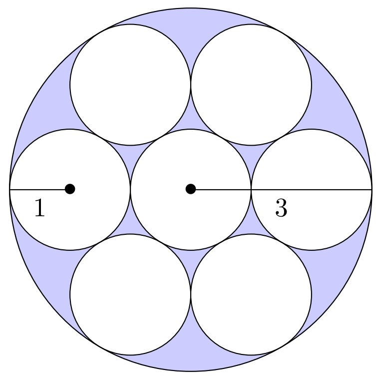
c) 2π

d) 3π

e) 3.5π

Answer: c)

Solution) The outer larger circle has radius 3 as shown in the figure below



and so we find that

$$\begin{aligned}\text{(the area of the shaded region)} &= \text{(the area of the larger circle)} - \text{(the areas of the 7 smaller circles)}\\ &= \pi \cdot 3^2 - 7 \cdot (\pi \cdot 1^2)\\ &= 2\pi.\end{aligned}$$

7. If x, y are real numbers satisfying $x^2 + y^2 = 2$, what is the maximum value of $2x + 2y$?

a) 1

b) 2

c) 3

d) 4

e) 5

Answer: d)

Solution I) Note that

$$\begin{aligned}0 \leq (x-1)^2 + (y-1)^2 &= (x^2 - 2x + 1) + (y^2 - 2y + 1)\\ &= (x^2 + y^2) - (2x + 2y) + 2\\ &= 4 - (2x + 2y)\end{aligned}$$

and so $2x + 2y \leq 4$ with equality when $x = 1$ and $y = 1$.

Solution II) Set $x = \sqrt{2}\cos\theta$ and $y = \sqrt{2}\sin\theta$ so that $x^2 + y^2 = 2$. The function

$$2x + 2y = 2\sqrt{2}\cos\theta + 2\sqrt{2}\sin\theta = 4\sin\left(\theta + \frac{\pi}{4}\right)$$

takes the maximum value when $\sin\left(\theta + \frac{\pi}{4}\right) = 1$.

8. Let f be a function with the following properties:
 i) $f(1) = 1$ and
 ii) $f(2n) = n \cdot f(n)$ for any positive integer n.
What is the value of $f(2^{10})$?

a) 1

b) 2^{10}

c) 2^{25}

d) 2^{30}

e) 2^{45}

Answer: e)

Solution) Using the property ii), we find that

$$\begin{aligned} f(2^{10}) &= f(2 \cdot 2^9) \\ &= 2^9 \cdot f(2^9) \\ &= 2^9 \cdot f(2 \cdot 2^8) \\ &= 2^9 \cdot 2^8 \cdot f(2^8) \\ &= 2^{9+8} \cdot f(2^8) \\ &= \vdots \\ &= 2^{9+8+\cdots+2+1+0} \cdot f(2^0) \\ &= 2^{9+8+\cdots+2+1+0} \cdot 1 = 2^{\frac{9 \cdot 10}{2}} = 2^{45}. \end{aligned}$$

9. How many integers between 1 and 2021 (inclusive) share a prime factor with 10?

a) 404

b) 607

c) 809

d) 1212

e) 1414

Answer: d)

Solution) Observe that

- mutiples of 2 share a common factor 2 with $10 = 2 \cdot 5$ and there are $\lfloor \frac{2021}{2} \rfloor = 1010$ multiples of 2 between 1 and 2021.
- mutiples of 5 share a common factor 5 with $10 = 2 \cdot 5$ and there are $\lfloor \frac{2021}{5} \rfloor = 404$ multiples of 5 between 1 and 2021.
- mutiples of 10 share a common factor 10 with $10 = 2 \cdot 5$ and there are $\lfloor \frac{2021}{10} \rfloor = 202$ multiples of 10 between 1 and 2021.
- integers not having factor a factor 2 or 5 do not share a common factor with 10.

By the inclusion-exclusion principle, there are

$$1010 + 404 - 202 = 1212$$

integers sharing a common factor with 10.

10. Fairies on the planet Twinkle Star have an alphabet with only the three letters A, B, C. They want to make licence plates for flying cars that use four letters $\boxed{}\boxed{}\boxed{}\boxed{}$. (For example, $\boxed{A}\boxed{B}\boxed{C}\boxed{C}$ and $\boxed{C}\boxed{C}\boxed{A}\boxed{B}$ are possible plates and they are considered different.) How many plates can they make?

a) 3

b) 9

c) 15

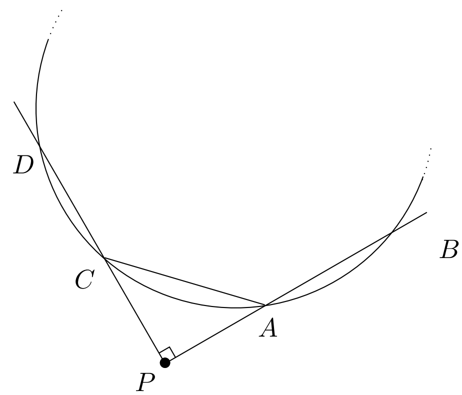
d) 27

e) 81

Answer: e).

Solution) We can place any one of three letters in the first blank, any one of three letters in the second, third and fourth blanks. There are total of $3 \cdot 3 \cdot 3 \cdot 3 = 81$ plates.

11. Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines which intersect a circle as shown in the figure below



Furthermore, suppose that the two lines intersect at a right angle at a point P outside the circle, $PA = 4$, $AB = 6$ and $CD = 3$. Find the length AC.

a) 6

b) $\sqrt{41}$

c) $\sqrt{47}$

d) 7

e) $\sqrt{53}$

Answer: b)

Solution) Using the two secant theorem, we find that[1]

$$PA \cdot PB = PC \cdot PD \implies 4 \cdot 10 = PC(PC + 3) \implies PC = 5.$$

Using the Pythagorean theorem to the right triangle $\triangle APC$, we find that

$$AC^2 = AP^2 + PC^2 = 4^2 + 5^2 \implies AP = \sqrt{41}.$$

12. What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

a) 20

b) 25

[1] Let $x = PC$. The middle equation becomes

$$40 = x^2 + 3x \implies (x - 5)(x + 8) = 0 \implies x = 5 > 0.$$

c) 10

d) 30

e) 45

Answer: a)

Solution) Let $y = \sqrt{x^2 + 18x + 45}$. Then

$$x^2 + 18x + 30 = (x^2 + 18x + 45) - 15 = y - 15$$

and so the original equation becomes

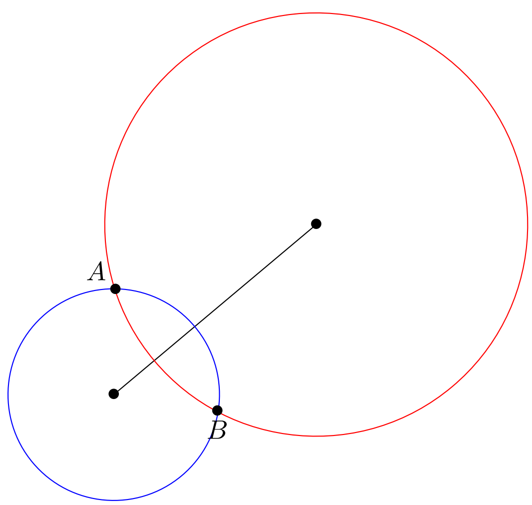
$$y^2 - 15 = 2y \implies (y-5)(y+3) = 0 \Rightarrow y = 5, -3.$$

Since $y \geq 0$, we have $y = 5$. Thus

$$(*) \quad \sqrt{x^2 + 18x + 45} = 5 \implies x^2 + 18x + 20 = 0.$$

Since $D = 18^2 - 4 \cdot 20 > 0$, the two roots[2] of this quadraric polynomial are real, and by Vieta's formula, their product is 20.

13. Suppose that two cicles of radii 7 and 13 intersect at the points A and B.



If $AB = 10$, what is the distance between the center of the two circles?

[2]There are no extraneous solution to $(*)$ because

$$x^2 + 18x + 20 = 0 \implies \sqrt{x^2 + 18x + 45} = \sqrt{(x^2 + 18x + 20) + 25} = \sqrt{25} = 5.$$

a) $2(\sqrt{6}+6)$

b) $3(\sqrt{7}+1)$

c) 16

d) 20

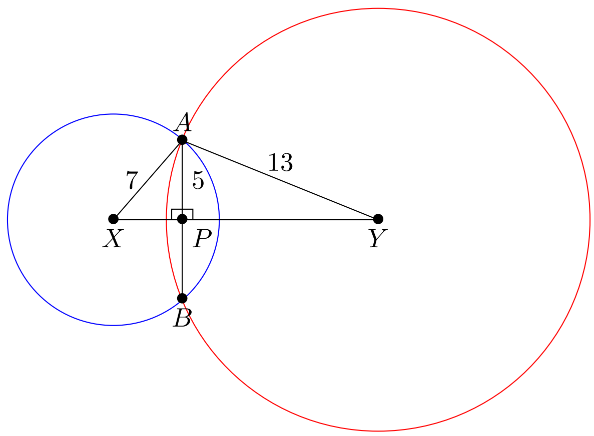
e) $4(\sqrt{11}+1)$

Answer: a)

Solution) Let X be the center of the first circle, Y the center of the second circle and P the point of intersection between $\overline{XY}$ and $\overline{AB}$. By the perpendicular bisector theorem together with $AB = 10$, we find that

$$AP = BP \implies AP = BP = 5.$$

and $\overline{XY} \perp \overline{AB}$.



Applying the Pythagorean theorem to the right triangle $\triangle XPA$, we find that

$$XP^2 = AX^2 - AP^2 = 7^2 - 5^2 = 24 \implies XP = 2\sqrt{6}.$$

Applying the Pythagorean theorem to the right triangle $\triangle YPA$, we find that

$$YP^2 = AY^2 - AP^2 = 13^2 - 5^2 = 144 \implies YP = 12.$$

The distance between the centers of the two circles is

$$XY = XP + YP = 2\sqrt{6} + 12 = 2(6+\sqrt{6}).$$

14. The sum of 18 consecutive positive integers is a perfect square. (A perfect square is an integer square such as $4 = 2^2$ or $25 = 5^2$.) The smallest possible value of this sum is

a) 169

b) 225

c) 289

d) 361

e) 441

Answer: b)

Solution I) The sum of 18 consecutive positive integers starting with a positive a is

$$S = a+(a+1)+(a+2)+\cdots+(a+16)+(a+17) = \frac{18\cdot(a+(a+17))}{2} = 9\cdot(2a+17)$$

For the sum $S = 9\cdot(2a+17) = 3^2\cdot(2a+17)$ to be a perfect square, $2a+17$ must be a perfect square. Checking for $a = 1, 2, 3, \cdots$

$$\begin{cases} a=1 & \Longrightarrow 2a+17 = 19 \text{ is not a perfect square} \\ a=2 & \Longrightarrow 2a+17 = 21 \text{ is not a perfect square} \\ a=3 & \Longrightarrow 2a+17 = 23 \text{ is not a perfect square} \\ a=4 & \Longrightarrow 2a+17 = 25 = 5^2 \text{ is a perfect square} \end{cases}$$

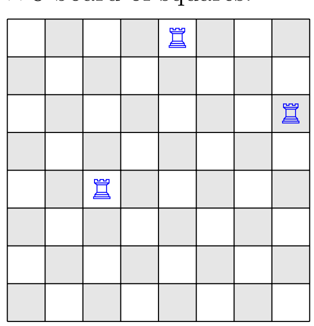
We find that the smallest such a sum is $S = 9\cdot(2a+17) = 9\cdot 25 = 225$ when $a = 4$.

Solution II) We check out several cases (the method of trial and error)

$$\begin{aligned} 1+2+\cdots+17+18 &= (18\cdot 19)/2 = 171 \text{ is not a perfect square} \\ 2+3+\cdots+18+19 &= (18\cdot 21)/2 = 189 \text{ is not a perfect square} \\ 3+4+\cdots+19+20 &= (18\cdot 23)/2 = 207 \text{ is not a perfect square} \\ 4+5+\cdots+20+21 &= (18\cdot 25)/2 = 225 = 15^2 \text{ is a perfect square} \end{aligned}$$

and we find 225 is the smallest such a sum.

15. Chess is played on an 8×8 board of squares.



How many ways are there to put 3 identical rooks on the chessboard so that no two rooks are in the same row or column?

a) 56

b) 336

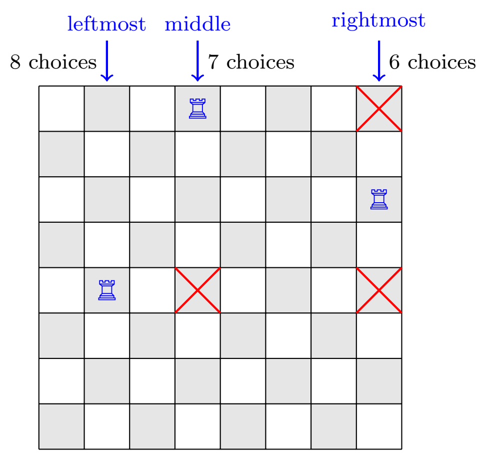
c) 6720

d) 18816

e) 112896

Answer: d)

Solution I) First, determine 3 columns for the rooks to be place. There are $\binom{8}{3} = 56$ ways to choose 3 columns. Once the columns are determined, there are 8 ways to choose a row for the leftmost column, 7 ways to choose a row for the the middle column and 6 ways to choose a row for the rightmost column.



There are a total of $56 \cdot 8 \cdot 7 \cdot 6 = 18816$ options.

Solution II) Another way is to place the rooks in order so they cannot capture, leading to $\binom{8}{1}\binom{8}{1} = 64$, $\binom{7}{1}\binom{7}{1} = 49$ and $\binom{6}{1}\binom{6}{1} = 36$ possibilities, then divide their product $64 \cdot 49 \cdot 36$ by $3! = 6$ to account for the reorderings. The answer is

$$\frac{64 \cdot 49 \cdot 36}{6} = 18816.$$

16. How many prime numbers are there between $2021! + 2$ and $2021! + 2021$?

(Here, 2021! is the product of all natural numbers between 1 and 2021, i.e.,

$$2021! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2017 \cdot 2018 \cdot 2019 \cdot 2020 \cdot 2021 \qquad).$$

a) 0

b) 1

c) 5

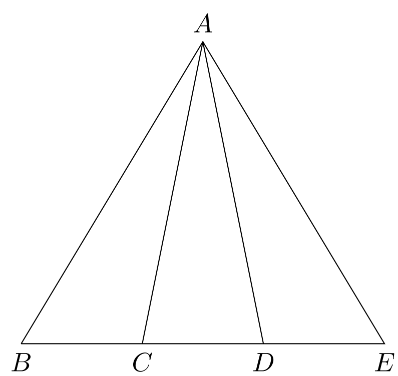
d) 19

e) 35

Answer: a).

Solution) For each natural number n for $2 \leq n \leq 2021$, the number $2021! + n$ is divisible by n (e.g., the number $2021!+2$ is divisible by 2, the number $2021!+3$ is divisible by 3 and so on) and so $2021! + n$ is not prime number. There is no prime numbers between $2021! + 2$ and $2021! + 2021$.

17. Suppose that we have points arranged as in the following figure:

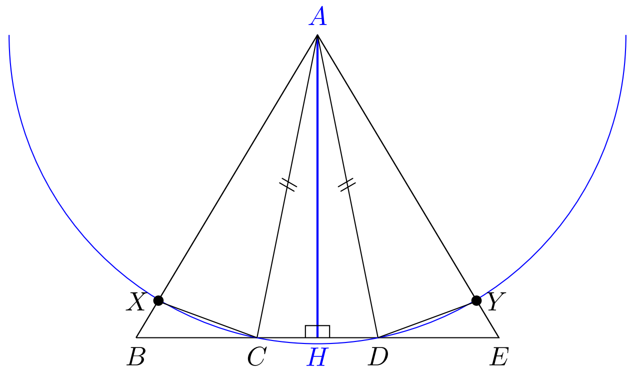


If $AB = AE$ and $BC = CD = DE$, which of the following is true?

a) $\angle BAC = \angle EAD = \angle CAD$.
b) $\angle BAC = \angle EAD < \angle CAD$
c) $\angle BAC = \angle EAD > \angle CAD$
d) $\angle BAC > \angle EAD > \angle CAD$
e) $\angle EAD > \angle BAC > \angle CAD$

Answer: b)

Solution) Let H be the point bisectiong $\overline{BE}$.



The perpendicular bisector theorem implies that $\triangle ACD$ is isosceles. The figure above is symmetric with respect $\overline{AH}$ and so

$$(*) \quad \angle BAC = \angle EAD.$$

Note that

$$\angle ACD < 90^\circ \implies \angle ACB = 180^\circ - ACD > 90^\circ.$$

By the scalene inequality[3],

$$(\angle ACB) > (\angle ABC) \implies AB > AC$$

which implies that the circle with radius $AC = AD$ with center at A will meet at X, Y in the interior of $\overline{AB}$ and $\overline{AE}$, respectively. Since $\angle AXC < 90^\circ, \angle BXC > 90^\circ > \angle XBC$ and, by the scalene inequality again,

$$\begin{cases} XC < BC \\ BC = CD \end{cases} \implies XC < CD \implies \angle XAC < \angle CAD$$

we conclude that

$$(**) \quad \angle BAC = \angle XAC < \angle CAD.$$

Combining $(*)$ and $(**)$ we find that $\angle BAC = \angle EAD < \angle CAD$.

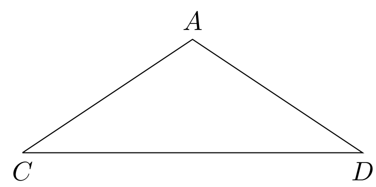
Solution II) The problem can be solved through a process of elimination. First, let H be the midpoint of the segment $\overline{BE}$. By the perpendicular bisector theorem since $AB = AE$ the line AH will be perpendicular to the line BE. Therefore if we reflect over the line AH, the angle $\angle CAH$ will be sent to the angle $\angle DAH$, the angle $\angle BAH$ will be sent to the angle $\angle EAH$ and therefore:

$$\angle BAC = \angle BAH - \angle CAH = \angle EAH - \angle DAH = \angle DAE$$

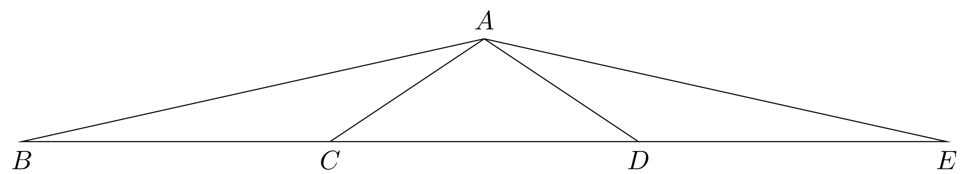
This eliminates the last two answers.

Now it only needs to be determined if the middle angle is larger, smaller, or equal in size to the outer angles. There are a variety of ways to see it must be larger. One way is to consider the case where we start by drawing an isosceles triangle $\triangle CAD$ with obtuse angle at A:

[3]In a single triangle, the larger side is opposite to the larger angle.

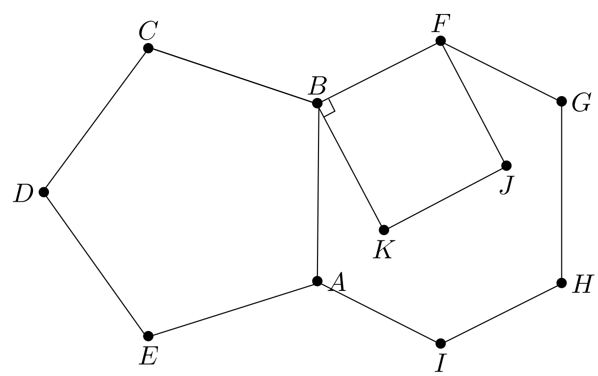


Now suppose that we choose points B and E to either side of C and D in such a way that $BC = CD = DE$:



Then we end with a situation meeting the conditions of the problem. Since $\angle BAE < 180°$ and $\angle CAD > 90°$ (since we made it obtuse), it follows that $\angle CAB$ and $\angle DAE$ must both be acute (in fact, both are less than 45 degrees). While this is not a rigorous proof that the middle angle is *always* larger, it does allow us to say that the only possible answer is b).

18. A regular pentagon $ABCDE$ and a regular hexgon $ABFGHI$ share a side AB and a square $BFJK$ is placed in the interior of the hexagon. What is $\angle CBK$?



a) 120°
b) 134°
c) 136°
d) 138°
e) 140°

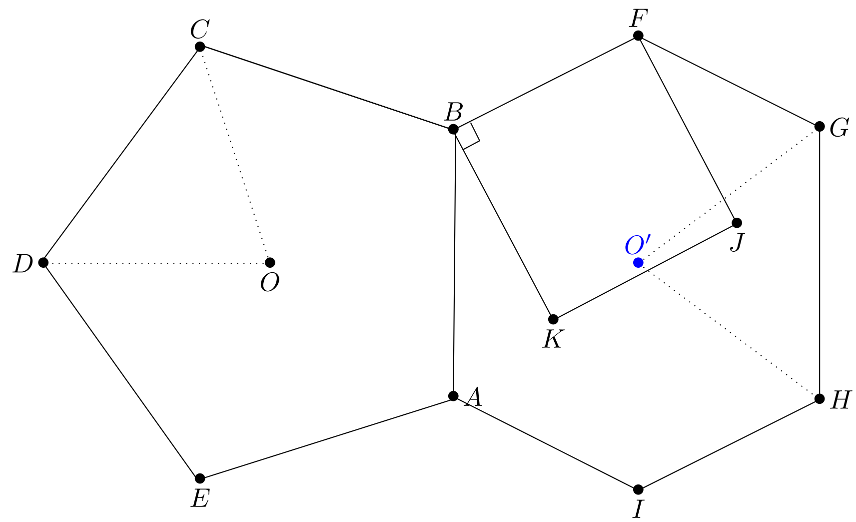
Answer: d)

Solution) Considering the rotational symmetry of the regular pentagon, we find that

$$\angle OCD = \frac{360^\circ}{5} = 72^\circ \implies \angle ABC = 180^\circ - 72^\circ = 108^\circ.$$

Similarly, considering the rotational symmetry of the regular hexgon, we find that

$$\angle O'GH = \frac{360^\circ}{6} = 60^\circ \implies \angle ABF = 180^\circ - 60^\circ = 120^\circ.$$



Since $\square BFJK$ is a square, $\angle KBF = 90^\circ$ and

$$\angle ABK = \angle ABF - \angle KBF = 120^\circ - 90^\circ = 30^\circ.$$

Finally, we find

$$\angle KBC = \angle ABK + \angle ABC = 30^\circ + 108^\circ = 138^\circ.$$

19. If θ is a solution of the equation

$$\sin(195^\circ) + \sin(105^\circ) = \sin\theta$$

then $\cos(2\theta)$ is

a) -1

b) $-\frac{1}{2}$

c) $-\frac{\sqrt{3}}{2}$

d) 0

e) $-\frac{\sqrt{6}}{2}$

Answer: d)

Solution) By sum-to-product formula

$$\begin{aligned}\sin\theta = \sin(195^\circ) + \sin(105^\circ) &= 2\sin\left(\frac{195^\circ + 105^\circ}{2}\right)\cos\left(\frac{195^\circ - 105^\circ}{2}\right)\\ &= 2\sin(150^\circ)\cos(45^\circ)\\ &= 2\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Now we use the double angle formula

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ &= 1 - 2\cdot\frac{1}{2} = 0\end{aligned}$$

Alternatively we may solve

$$\sin\theta = \frac{\sqrt{2}}{2} \implies \begin{cases}\theta = 45^\circ + k\cdot 360^\circ & \text{or}\\ \theta = 135^\circ + k\cdot 360^\circ\end{cases} \implies \begin{cases}2\theta = 90^\circ + 2k\cdot 360^\circ & \text{or}\\ 2\theta = 270^\circ + 2k\cdot 360^\circ\end{cases}$$

where k is an integer, and so $\cos(2\theta) = 0$.

20. How many real numbers x satisfy

$$\sin x = \frac{x}{6\pi} \quad ?$$

a) 9

b) 11

c) 13

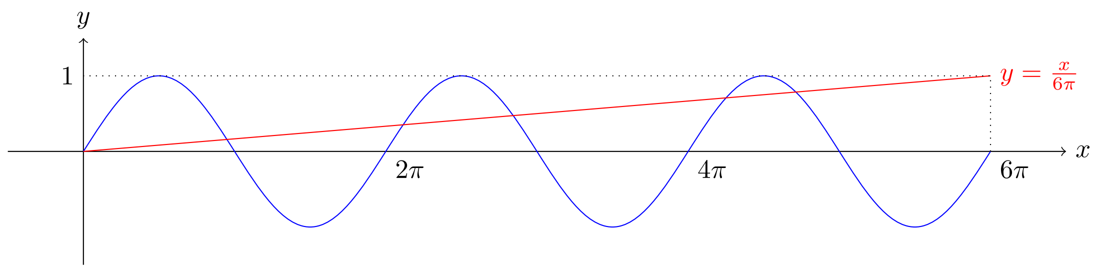
d) 15

e) 16

Answer: b)

Solution) Clearly $x = 0$ is a solution. Also, if $x > 0$ is a solution, so is $-x < 0$. So let's first consider positive solutions.

If x is a positive solution, then $0 < x = 6\pi\sin x \leq 6\pi$. Considering the graph of $y = \sin x$ and $y = \frac{x}{6\pi}$ over the interval $(0, 6\pi)$



we find 5 positive solutions to the equation $\sin x = \frac{x}{6\pi}$. Similary, we find 5 negative solutions to the equation. There are $5 + 1 + 5 = 11$ solutions in total.

21. In each cell of the following 3×3 table, fill in 1 or -1, such that the product of numbers in each row or column is equal to 1. For example,

fill in $\longrightarrow$

-1	1	-1
1	1	1
-1	1	-1

How many ways are there to fill in this table?

a) 12

b) 16

c) 24

d) 36

e) 18

Answer: b)

Solution) We first fill the top left 2×2 square Q at will. Since there are $2 \times 2 = 4$ cells, and each cell can take 1 or -1 value, there are $2^4 = 16$ ways to fill it.

a	b	
c	d	

We will show how to determine x on the top right corner

a	b	x
c	d	

Keeping in mind that a, b are either 1 or -1 and the product of the first row is 1, we find that

$$abx = 1 \implies x = ab.$$

Similarly, we can determine the remaining entries except the bottom right one $(*)$ uniquely as follows

a	b	ab
c	d	cd
ab	bd	$(*)$

The last entry $(*)$ is uniquely and consistently determined as well because

$$(ac)(bd) = (ab)(cd).$$

Thus, there are $2^4 = 16$ ways to fill this table.

22. The remainder of $3^{21} + 5^{21}$ divided by 64 is

a) 4

b) 20

c) 12

d) 10

e) 40

Answer: e)

Solution) After applyin the binormial theorem we work in modulo 64 arithmetic

$$\begin{aligned}
3^{21}+5^{21} &= (4-1)^{21}+(4+1)^{21} \\
&= \left[\sum_{k=0}^{21}\binom{21}{k}4^k\cdot(-1)^{21-k}\right]+\left[\sum_{k=0}^{21}\binom{21}{k}4^k\cdot 1^{21-k}\right] \\
&= \sum_{k=0}^{21}\binom{21}{k}4^k\cdot[(-1)^{21-k}+1^{21-k}] \\
&\equiv \sum_{k=0}^{2}\binom{21}{k}4^k\cdot[(-1)^{21-k}+1^{21-k}] \qquad \text{(because } 4^k\equiv 0 \mod 64 \text{ for } k\geq 3) \\
&\equiv \binom{21}{0}4^0\cdot[(-1)^{21}+1^{21}]+\binom{21}{1}4^1\cdot[(-1)^{20}+1^{20}]+\binom{21}{2}4^2\cdot[(-1)^{19}+1^{19}] \\
&\equiv \binom{21}{1}4^1\cdot[(-1)^{20}+1^{20}] \\
&\equiv 21\cdot 4\cdot 2\equiv 40 \mod 64.
\end{aligned}$$

23. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walters entered?

a) 71

b) 76

c) 80

d) 82

e) 91

Answer: c)

Solution I) Let x_1, x_2, x_3, x_4, x_5 be the scores entered in that order. The sequence of averages

$$x_1, \quad \frac{x_1+x_2}{2}, \quad \frac{x_1+x_2+x_3}{3}, \quad \frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{x_1+x_2+x_3+x_4+x_5}{5}$$

are all integers. From this, we find that

i) x_1 is an integer
ii) x_1+x_2 is divisible by 2
iii) $x_1+x_2+x_3$ is divisible by 3
iv) $x_1+x_2+x_3+x_4$ is divisible by 4

v) $x_1 + x_2 + x_3 + x_4 + x_5 = 71 + 76 + 80 + 82 + 91 = 400$.

Since $x_1 + x_2 + x_3 + x_4$ is divisible by 4 and 400 is divisible by 4, x_5 must also be divisible by 4. Therefore $x_5 = 76$ or $x_4 = 80$.

Case I) Assume $x_5 = 76$. Then $x_1 + x_2 + x_3 + x_4 = 324$ and so

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 324 \equiv 0 \mod 3 \\ x_1 + x_2 + x_3 \equiv 0 \mod 3 \end{cases} \implies x_4 \equiv 0 \mod 3$$

But none of $71, 80, 82, 91$ is divisible by 3. $x_5 = 76$ is not possible.

Case II) Assume $x_5 = 80$. Then $x_1 + x_2 + x_3 + x_4 = 320$ and so

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 320 \equiv 2 \mod 3 \\ x_1 + x_2 + x_3 \equiv 0 \mod 3 \end{cases} \implies x_4 \equiv 2 \mod 3$$

Since 71 is the only number congruent to 2 modulo 3 among $71, 76, 82, 91$ is congruent to 2 modulo 3 we find that $x_4 = 71$ and $x_1 + x_2 + x_3 = 249$.

$$\begin{cases} x_1 + x_2 + x_3 = 249 \equiv 1 \mod 2 \\ x_1 + x_2 \equiv 0 \mod 2 \end{cases} \implies x_3 \equiv 1 \mod 2$$

Since 91 is the only odd number among $76, 82, 91$ and so $x_3 = 91$. There are two possible cases

$$(x_1, x_2, x_3, x_4, x_5) = (76, 82, 91, 71, 80)$$

or

$$(x_1, x_2, x_3, x_4, x_5) = (82, 76, 91, 71, 80).$$

In any case, $x_5 = 80$.

Solution II) Let x_1, x_2, x_3, x_4, x_5 be the scores entered in that order. The sequence of averages

$$x_1, \quad \frac{x_1 + x_2}{2}, \quad \frac{x_1 + x_2 + x_3}{3}, \quad \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$$

are all integers. From this, we find that

i) x_1 is an integer
ii) $x_1 + x_2$ is divisible by 2
iii) $x_1 + x_2 + x_3$ is divisible by 3
iv) $x_1 + x_2 + x_3 + x_4$ is divisible by 4
v) $x_1 + x_2 + x_3 + x_4 + x_5 = 71 + 76 + 80 + 82 + 91 = 400$.

Since

$$x_1 + x_2 + x_3 \equiv 0 \mod 3$$

and

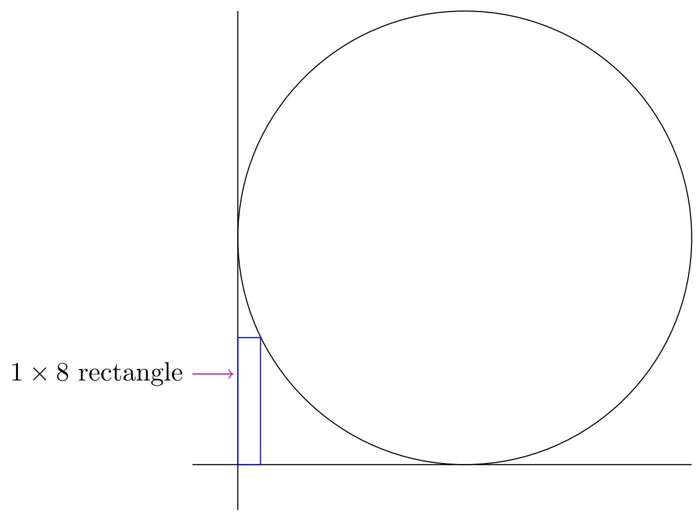
$$(71, 76, 80, 82, 91) \equiv (2, 1, 2, 1, 1) \mod 3$$

the only way to get a number divisible by 3 by adding three of these is by adding the three ones and so $76, 82, 91$ must go first and so $x_1 + x_2 + x_3 = 76 + 82 + 91 = 249$ and x_4, x_5 are shuffles of $71, 80$ From

$$\begin{cases} x_1 + x_2 + x_3 = 249 \equiv 1 \mod 4 \\ x_1 + x_2 + x_3 + x_4 \equiv 0 \mod 4. \end{cases} \implies x_4 \equiv -1 \mod 4 \implies x_4 = 71$$

and $x_5 = 80$.

24. Suppose that a circle is tangent to two perpendicular lines so that a 1×8 rectangle can be placed between the perpendicular lines and the circle as in the following figure



What is the radius of the circle?

a) 7

b) 9

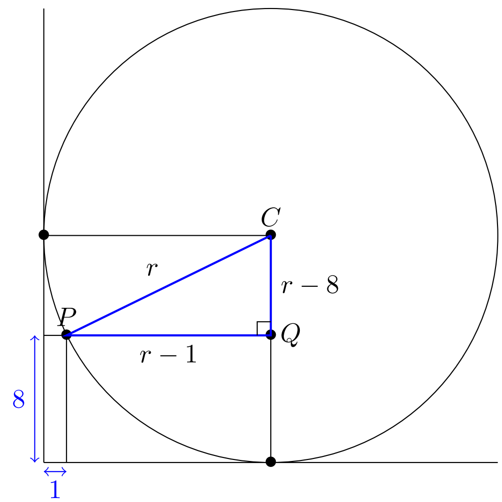
c) 12

d) 13

e) 16

Answer: d)

Solution) Let r be the radius of the circle centered at C. Draw a right triangle $\triangle CQP$ as shown below



Applying the Pythagorean theorem to $\triangle CQP$ we find that

$$(r-1)^2 + (r-8)^2 = r^2 \implies r^2 - 18r + 65 = 0 \implies r = 5, 13$$

The $r = 5$ is not possible by considering the geometry shape above. The radius is $r = 13$.

25. Suppose $x \neq 1$. The product of real numbers x satisfying

$$\log_2 x - 1 - \log_x 4 = 0$$

is

a) 8

b) 1

c) 2

d) 6

e) 3

Answer: c)

Solution) Let $u = \log_2 x$. Then $u \neq 0$ because $x \neq 1$. By the base change formula

$$\log_x 4 = \frac{\log_2 4}{\log_2 x} = \frac{2}{u}$$

so the given equation becomes

$$u - 1 - \frac{2}{u} = 0 \implies u^2 - u - 2 = 0 \implies u = -1, 2$$

and

$$\begin{cases} u = \log_2 x = -1 \implies x = 2^{-1} = \frac{1}{2} \\ u = \log_2 x = 2 \implies x = 2^2 = 4 \end{cases}$$

Hence the product of all such x is $4 \cdot \frac{1}{2} = 2$.

26. Let $f(x)$ denote the sum of the digits of the positive integer x. For example, $f(8) = 8$ and $f(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $f(f(x)) = 3$?

a) 3

b) 4

c) 6

d) 9

e) 10

Answer: e)

Solution) Let $y = f(x)$. Then $y \leq 18$ because $x \leq 99$ and

$$f(f(x)) = 2 \implies f(y) = 3 \implies y = 3 \text{ or } y = 12$$

because, for $0 \leq a \leq 9$ and $0 \leq b \leq 9$,

$$(*) \quad y = 10a + b \leq 18 \implies \begin{cases} a = 1 \implies y = 10 + b \implies f(y) = 1 + b = 3 \implies b = 2 \\ a = 0 \implies y = 0 + b \implies f(y) = b = 3 \implies b = 3 \end{cases}$$

Thus if $f(y) = 3$, then $y = f(x) = 3$ or $y = f(x) = 12$. Using the computation similar to $(*)$, we find that

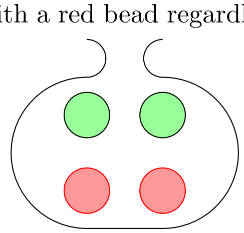
$$f(x) = 3 \implies x = 12, 21, 30$$

and

$$f(x) = 12 \implies x = 39, 48, 57, 66, 75, 84, 93.$$

There are 10 solutions x to $f(f(x)) = 3$.

27. A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out.



What is the probability that all beads in the bag are red after three such replacements?

a) 1/8

b) 5/32

c) 9/32

d) 3/8

e) 7/16

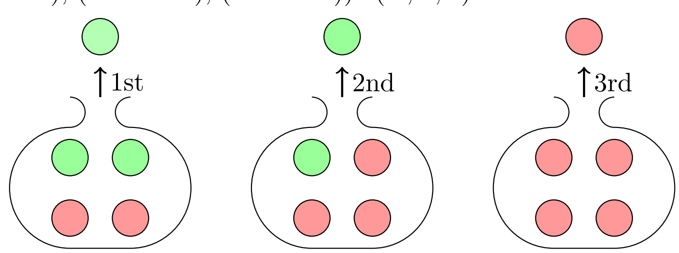
Answer: c)

Solution) Let R be the red bead and G be the green bead and think of triple

$$((\text{1st bead}), (\text{2nd bead}), (\text{3rd bead})).$$

There are three possible cases:

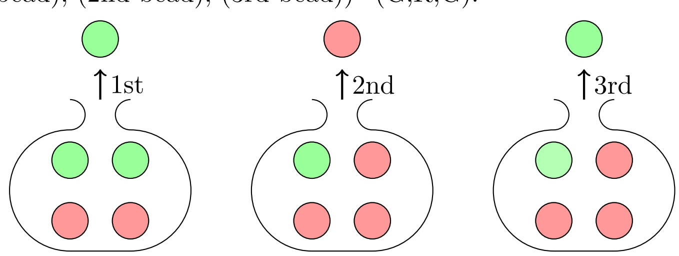
Case I) ((1st bead), (2nd bead), (3rd bead))=(G,G,R).



The probability for this case is

$$\frac{2}{4} \times \frac{1}{4} \times \frac{4}{4} = \frac{1}{8}.$$

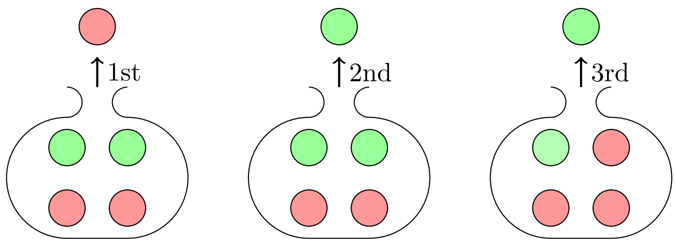
Case II) ((1st bead), (2nd bead), (3rd bead))=(G,R,G).



The probability for this case is

$$\frac{2}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{32}.$$

Case III) ((1st bead), (2nd bead), (3rd bead))=(R,G,G).



The third case is in which the green beads are chosen second and third. The probability for this case is

$$\frac{2}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Adding the probabilities of the three cases gives the desired probability

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} = \frac{9}{32}.$$

28. Let $f(x)$ be a function not defined for $x = 0$ but

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \qquad \text{for all nonzero } x.$$

Find the solutions of the equation

$$f(x) = f(-x).$$

a) $x = \pm 1$

b) $x = \pm\sqrt{2}$

c) $x = \pm\sqrt{3}$

d) $x = \pm\sqrt{5}$

e) $x = \pm\sqrt{7}$

Answer: b)

Solution) Replacing x by $\frac{1}{x}$ in

$$(*) \qquad f(x) + 2f\left(\frac{1}{x}\right) = 3x$$

we obtain that

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \implies (**) \qquad 2f\left(\frac{1}{x}\right) + 4f(x) = \frac{6}{x}.$$

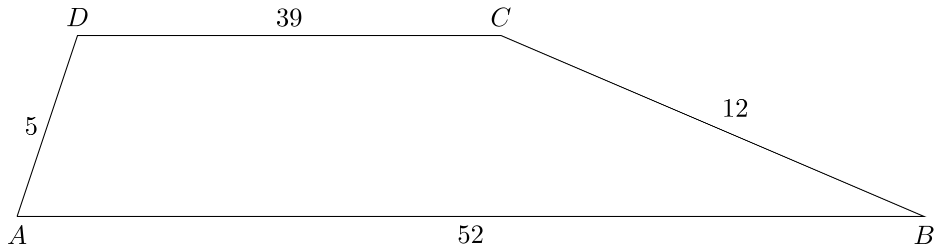
Subtracting $(*)$ from $(**)$, we find that

$$3f(x) = \frac{6}{x} - 3x \implies f(x) = \frac{2}{x} - x.$$

The solutions of the equation

$$f(x) = f(-x) \implies \frac{2}{x} - x = -\frac{2}{x} + x \implies x^2 = 2 \implies x = \pm\sqrt{2}.$$

29. In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$.



The area of $ABCD$ is

a) 182

b) 195

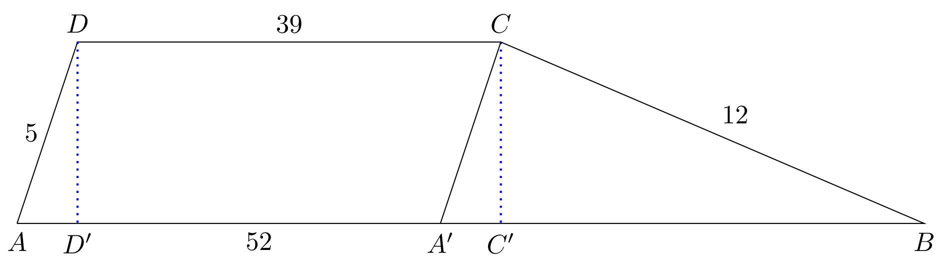
c) 210

d) 234

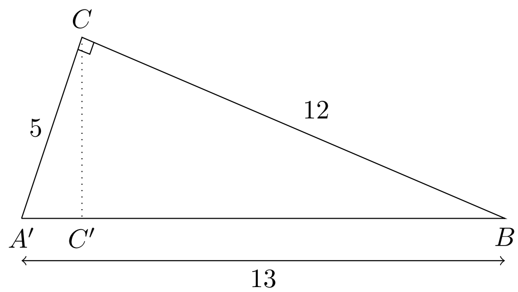
e) 260

Answer: c)

Solution) Draw points C' and D' on the segment $\overline{AB}$ so that CC' and DD' be the altitude of the trapezoid as show below.



Move the triangle $\Delta ADD'$ so that DD' coincides with CC' and let $\Delta A'CC'$ be the resulting trianlge.



Considering angles we find that $\angle A'CB = 90°$ and so $\Delta A'CB$ becomes a right triangle with

$$\begin{aligned} A'B &= AB - AA' \\ &= AB - DC \\ &= 52 - 39 = 13. \end{aligned}$$

The length of $A'B$ in this triangle is equal to the length of the original AB, minus the length of CD. Thus $A'B = 52 - 39 = 13$. Computing the area of $\Delta A'CB$ in two different ways, we find that

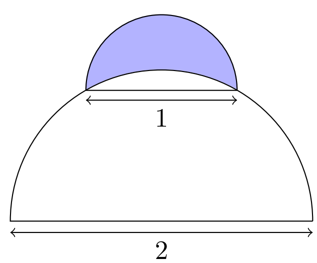
$$\frac{1}{2} \cdot 12 \cdot 5 = \frac{1}{2} \cdot 13 \cdot CC' \implies CC' = \frac{60}{13}.$$

The area of the trapezoid is

$$\frac{(AB + CD)}{2} \cdot CC' = \frac{(52 + 39)}{2} \cdot \frac{60}{13} = 210.$$

Remark) I changed the level of difficulty from Medium to Hard because I needed more hard problems and this problem is somewhat lengthy.

30. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2 as shown below.



The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune.

a) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

b) $\frac{\sqrt{3}}{4} - \frac{\pi}{12}$

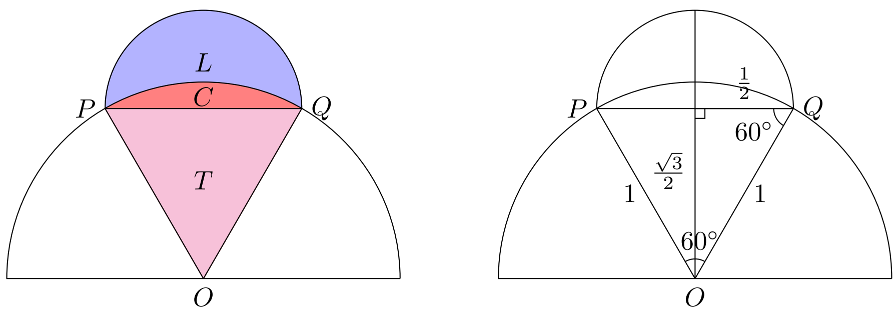
c) $\frac{\sqrt{3}}{4} - \frac{\pi}{24}$

d) $\frac{\sqrt{3}}{4} + \frac{\pi}{24}$

e) $\frac{\sqrt{3}}{4} + \frac{\pi}{12}$

Answer: c)

Solution) Let P and Q be the points of intersection of the smaller semicircle with the larger semicircle and let O be the center of the larger semicircle. Also, let L be the lune, let T be the triangle ΔOPQ and let C be the region in the circular sector formed by $\overset{\frown}{PQ}$ and O with the triangle T removed.



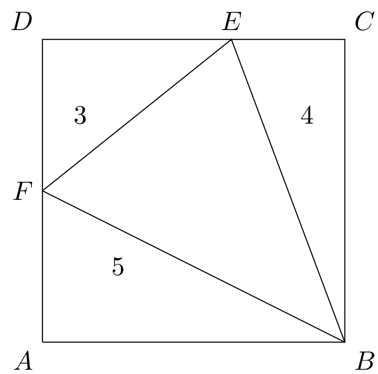
The area of the smaller semicircle is $\frac{1}{2} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{8}\pi$. Since the triangle T is an equilateral triangle with side length 1, the area of T is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. The angle of the arc $\overset{\frown}{PQ}$ is $60°$ and so the area of the circular sector determined by the angle $\angle POQ$ is

$$\pi \cdot 1^2 \cdot \frac{60}{360} = \frac{\pi}{6}.$$

Thinking of the areas, we find that

$$\begin{aligned}
(\text{the area of the lune } L) &= [(\text{the area of the smaller semicircle}) + (\text{the area of the triangle } T)] \\
&\quad - (\text{the area of the circular sector determined by } \angle POQ) \\
&= \left[\frac{1}{8}\pi + \frac{\sqrt{3}}{4}\right] - \frac{1}{6}\pi \\
&= \frac{\sqrt{3}}{4} - \frac{1}{24}\pi
\end{aligned}$$

Tie Breaker (TB) problems: In a square $ABCD$, the areas of $\triangle EDF$, $\triangle BCE$ and $\triangle FAB$ are 3, 4, and 5 respectively.



TB I) What is the area of $\triangle EFB$?

a) 6.5

b) 6

c) 5.5

d) 7

e) 8

TB II) What is the value of $\cos(\angle FEB)$?

a) $\frac{12}{\sqrt{1769}}$

b) 0

c) $\frac{13}{\sqrt{1769}}$

d) $\frac{14}{\sqrt{1769}}$

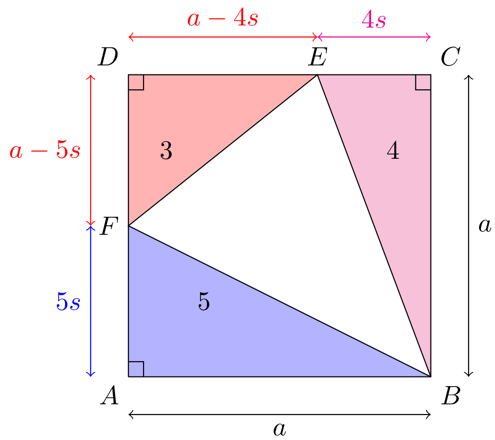
e) $\frac{12}{\sqrt{1767}}$

Answers: TB I: e) and TB II: c)

TB I Solution) Let a be the length of the side of the square. Considering the ratios of area of the triangles ΔBCE and ΔFAB

$$\text{(the area } \Delta BCE) : \text{(the area } \Delta BCE) = \frac{1}{2} \cdot a \cdot CE : \frac{1}{2} \cdot a \cdot AF = 4 : 5$$

we find that $CE : AF = 4 : 5$ and so we may write $CE = 4s$ and $AF = 5s$ for some $s > 0$.



Using the figure above, we find that $DE = a - 4s$ and $DF = a - 5s$. Note that

$$\text{(the area of } \Delta DEF) = \frac{1}{2} \cdot (a - 4s) \cdot (a - 5s) = 3$$

and

$$\text{(the area of } \Delta BCE) = \frac{1}{2} \cdot a \cdot (4s) = 4.$$

Taking the ratio of the two equations above we find that

$$\frac{(a-4s)(a-5s)}{4as} = \frac{3}{4} \implies a^2 - 12as + 20s^2 = 0 \implies (a-2s)(a-10s) = 0$$

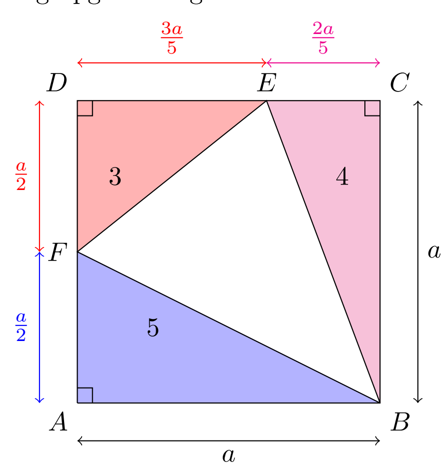
from which we get $s = \frac{a}{2}$ or $s = \frac{a}{10}$. We eliminate the possibility $s = \frac{a}{2}$ by observing that

$$5s = AF < AD = a \implies s < \frac{a}{5}.$$

Using $s = \frac{a}{10}$, we can express side lengths in terms of a only as follws

$$AF = 5s = \frac{a}{2} \qquad \text{and} \qquad CE = 4s = \frac{2a}{5}$$

and we have the following upgraded figure



and also find that

$$5 = (\text{the area of } \Delta ABF) = \frac{1}{2} \cdot a \cdot \frac{a}{2} \implies a^2 = 20 \implies a = 2\sqrt{5}.$$

On considering areas, we have

$$\begin{aligned}(\text{the area of } \Delta EFB) &= (\text{the area of } \Box ABCD)\\ &\quad - [(\text{the area of } \Delta FAB) + (\text{the area of } \Delta BCE) + (\text{the area of } \Delta EDF)]\\ &= a^2 - \Big[\frac{1}{2} \cdot a \cdot \frac{a}{2} + \frac{1}{2} \cdot a \cdot \frac{2a}{5} + \frac{1}{2} \cdot \frac{3a}{5} \cdot \frac{a}{2}\Big]\\ &= a^2 - \Big[\frac{a^2}{4} + \frac{a^2}{5} + \frac{3a^2}{20}\Big]\\ &= \frac{2}{5}a^2\\ &= \frac{2}{5} \cdot 20 = 8.\end{aligned}$$

TB II Solution) We use the 2nd law of cosines together with Pythagorean theorems to compute

$$\begin{aligned}\cos(\angle FEB &= \frac{EF^2 + BE^2 - BF^2}{2EF \cdot BE}\\ &= \frac{\big[(\frac{a}{2})^2 + (\frac{3a}{5})^2\big] + \big[a^2 + (\frac{2a}{5})^2\big] - \big[a^2 + (\frac{a}{2})^2\big]}{2 \cdot \sqrt{(\frac{a}{2})^2 + (\frac{3a}{5})^2} \cdot \sqrt{a^2 + (\frac{2a}{5})^2}}\\ &= \frac{13}{\sqrt{61}\sqrt{29}} = \frac{13}{\sqrt{1769}}.\end{aligned}$$