The 49th Annual High School Mathematics Contest
April 19, 2023
Minnesotata State University, Mankato
In memory of Dr. Waters who loved math and the math contest.

Instruction for the Contest

1. Write your answer on the **answer sheet**.
2. You have 90 minutes to work on 30 questions and one tie breaker problem.
3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.
4. You need to write a solution for the tie breaker problem on the **answer sheet**, and the tie breaker will be used only to break any possible ties that arise on the test.
5. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.
6. **Submit your answer sheet** at the end of the contest and keep the exam booklet.

1. Write $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ with a rational denominator. The result is

a) $\frac{3+2\sqrt{6}+\sqrt{60}}{12}$

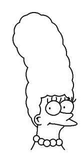
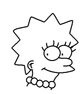
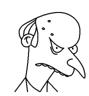
b) $\frac{3+2\sqrt{6}+\sqrt{15}}{6}$

c) $\frac{4+2\sqrt{6}+\sqrt{60}}{12}$

d) $\frac{3+\sqrt{6}+\sqrt{15}}{6}$

e) $\frac{3+\sqrt{6}+\sqrt{12}}{6}$

2. In Springfield where the following people live

 (Homer),  (Marge),  (Bart),  (Lisa),  (Mr. Burns)

a heated debate developed:

> Homer: Bart, the probability that you win in a math competition is 50% because either you win or you lose. Therefore, your chance of winning is 50%+50%=100% if you participate AMC[1] and MMC[2] math contests. Participate these contests and win!
>
> Marge: Homie, you are so smart.
>
> Bart: Cool! Dad, I will win one of the contests.
>
> Lisa: Dad! You are wrong.
>
> Mr. Burns: All of you are wrong!

Who is right?

a) Homer is right.

b) Marge is right.

c) Bart is right.

d) Lisa is right.

e) Mr. Burns is right.

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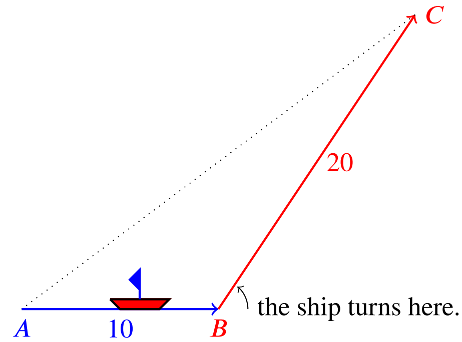
[1] American Mathematics Competitions

[2] Mankato Mathematics Competition

3. A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles.



Which of the following intervals contains AC^2 ?

a) $[400, 500)$

b) $[500, 600)$

c) $[600, 700)$

d) $[700, 800)$

e) $[800, 900)$

4. The function $f(x) = 2x + \frac{12}{x}$ is defined on the positive real numbers $x > 0$. At which value of x does $f(x) = 2x + \frac{12}{x}$ assume its minimum value?

a) $x = 1$

b) $x = \sqrt{2}$

c) $x = \sqrt{6}$

d) $x = \sqrt{12}$

e) $x = 6$

5. Let r, s, t be the roots of $x^3 - 6x^2 + 5x - 7 = 0$. Find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}.$$

a) $-\frac{59}{49}$

b) $-\frac{39}{29}$

c) $\frac{25}{49}$

d) $-\frac{16}{25}$

e) 3

6. If $2^{16^x} = 16^{2^x}$, then x is equal to

a) 2

b) $\frac{3}{4}$

c) $\frac{2}{3}$

d) $\frac{1}{3}$

e) 4

7. At the end of a shift, a store clerk notices that the average value of all the coins left in the till is 20 cents. Furthermore if there was one more quarter, the average value of all the coins would be 21 cents. How many pennies are in the till?

(The coins are all pennies, worth 1 cent, nickles, worth 5 cents, dimes, worth 10 cents, and quarters, worth 25 cents.)

a) 0

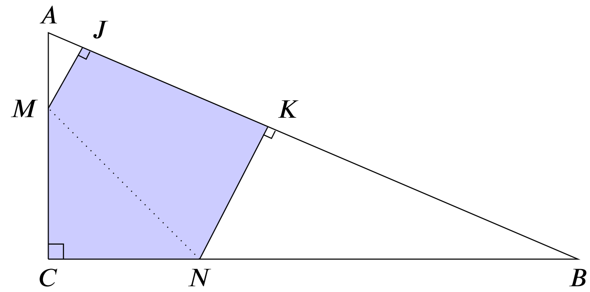
b) 1

c) 3

d) 5

e) 7

8. In $\triangle ABC$, $AB = 13$, $AC = 5$, and $BC = 12$. Points M and N lie on AC and BC, respectively, with $CM = CN = 4$. Points J and K are on AB so that MJ and NK are perpendicular to AB. What is the area of pentagon $CMJKN$?



(The right triangle ABC has area $1/2 \cdot AC \cdot BC = 1/2 \cdot 5 \cdot 12 = 30$).

a) 15

b) $\frac{81}{5}$

c) $\frac{205}{12}$

d) $\frac{240}{13}$

e) 20

9. Suppose that we have two cubes. The first cube has volume 8, and the second has twice the *surface area* of the first. What is the length of the sides of the second cube?

a) 2

b) $2\sqrt{2}$

c) 4

d) $3\sqrt{2}$

e) $3\sqrt[3]{4}$

10. Find the smallest positive integer n such that

$$\begin{cases} \text{the remainder is 2 if } n \text{ is divided by 3} \\ \text{the remainder is 3 if } n \text{ is divided by 5 and} \\ \text{the remainder is 2 if } n \text{ is divided by 7.} \end{cases}$$

a) 5

b) 18

c) 23

d) 128

e) 233

11. If $f\left(\frac{2x+1}{3x+2}\right) = x$ for all x except $-\frac{2}{3}$, determine $f(x)$.

a) $f(x) = \frac{3x+2}{2x+1}$

b) $f(x) = \frac{2x-1}{2-3x}$

c) $f(x) = \frac{3x-2}{2x-1}$

d) $f(x) = \frac{3x-2}{2x+1}$

e) $f(x) = \frac{2x-1}{3x-2}$.

12. The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the measures of the two angles?

a) 75°

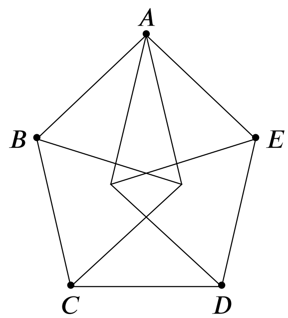
b) 90°

c) 135°

d) 150°

e) 270°

13. The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers.



What is $m + n$?

a) 23

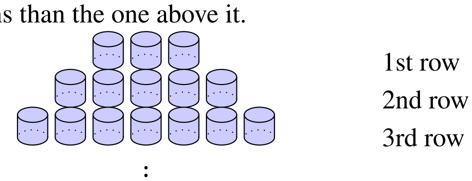
b) 24

c) 22

d) 20

e) 21

14. A store has a pile of cans arranged so that the top row has 3 cans and each subsequent row has two more cans than the one above it.



There are 120 cans in the whole pile. How many rows are in the pile?

a) 10

b) 11

c) 12

d) 13

e) 14

15. In the expansion of $\left(xy - 2y^{-3}\right)^{16}$, find the power of x in the term that does not contain y.

a) 12

b) 8

c) 6

d) 7

e) 10

16. The last digit of 7^{7^7} is

a) 7

b) 1

c) 3

d) 9

e) 5

17. Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, abc miles was displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a + b + c \leq 7$. At the end of the trip, the odometer showed cba miles. What is $a^2 + b^2 + c^2$?

a) 26

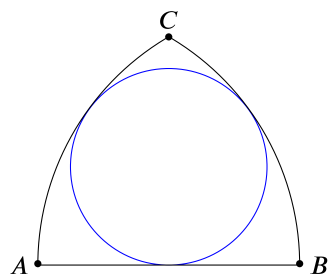
b) 27

c) 36

d) 37

e) 41

18. If circular arcs $\widehat{AC}$ and $\widehat{BC}$ have centers at B and A, respectively, then there exists a circle tangent to both $\widehat{AC}$ and $\widehat{BC}$, and to $\overline{AB}$.



If the length of the arc $\widehat{BC}$ is 4π, then the area of the circle is

a) $\frac{36\pi}{2}$

b) $\frac{81\pi}{4}$

c) $\frac{49\pi}{5}$

d) $\frac{67\pi}{6}$

e) $\frac{2023\pi}{49}$

19. Suppose that we choose an integer from 1 to 400 (inclusive), where we have equal probability of selecting any integer in the range. What is the probability that we select an even integer which has a 3 as one of its digits?

a) $53/400$

b) $3/20$

c) $13/80$

d) $33/200$

e) $3/16$

20. Let n be an integer. If the tens digit of n^2 is 7, what is the last digit of n^2?

a) 4

b) 1

c) 6

d) 9

e) 5

21. Suppose we expand the following product:

$$(1 + x + \cdots + x^{50})(1 + x + \cdots + x^{25})^2$$

What will be the coefficient in front of x^{10}?

a) 48

b) 55

c) 60

d) 66

e) 72

22. Suppose that Alice and Bob play the following game: They put a certain number of beads in a pile, which varies from game to game. They then take turns, starting with Alice. On each turn a player can take one or two beads. The player who takes the **last** bead wins. Both Alice and Bob are experts at their game and always make the best possible move. Which is the only possible true statement?

a) Alice won the games that started with 2, 8 and 22 beads.

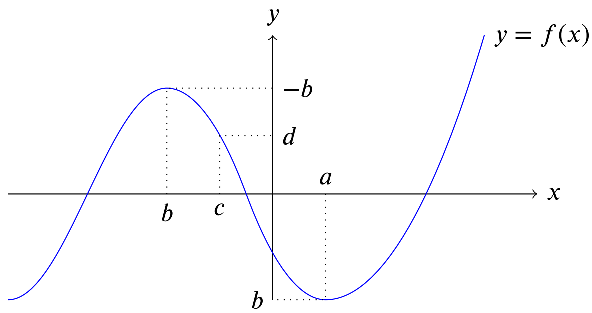
b) Bob won the games that started with 3, 7 and 12 beads.

c) Alice won the games that started with 1, 6 and 10 beads.

d) Bob won the games that started with 2, 5 and 16 beads.

e) Alice won the games that started with 1, 2 and 3 beads.

23. Consider the graph of a function $f : \mathbb{R} \to \mathbb{R}$ where a, b, c, d are real numbers.



i) $f(f(a)) = b$

ii) $f(a) + f(b) + f(c) > 0$

iii) $f(c) > c$

iv) $f\left(\frac{b+c}{2}\right) > \frac{f(b)+f(c)}{2}$

Choose all the correct statements.

a) None of them

b) ii), iii)

c) i), iii),iv)

d) ii), iii), iv)

e) All of them

24. Let f be a function defined by the following properties:

- $f(1) = 1$
- For any positive integer n we have $f(2n) = n \cdot f(n)$.

What is the value of $f(2^{50})$?

a) 2^{50}

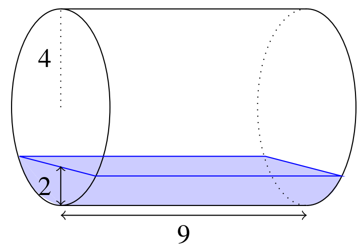
b) 2^{125}

c) 2^{624}

d) 2^{1225}

e) 2^{4950}

25. A cylinderical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet.



What is the volume of water, in cubic feet?

a) $24\pi - 36\sqrt{2}$

b) $24\pi - 24\sqrt{3}$

c) $36\pi - 36\sqrt{3}$

d) $36\pi - 24\sqrt{2}$

e) $48\pi - 36\sqrt{3}$

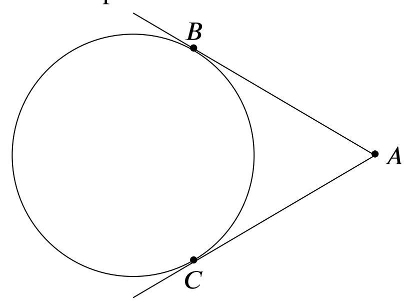
26. Detective Leblanc interviews three suspects, Agatha, Conan and Ronald. Each one of them is either a murder or an innocent suspect. The murderers all lied and the innocent suspects all told the truth. The following are the statements made by the suspects:

- Agatha: "Conan is the only murderer."
- Conan: "Agatha and Ronald are both innocent."
- Ronald: "Agatha and Conan are both murderers."

Which of the following is true?

a) Conan is the only murderer
b) Ronald is the only murderer
c) Agatha is the only innocent suspect.
d) Ronald is the only innocent suspect.
e) All three are murderers.

27. Two tangents to a circle intersect each other at a point A outside of the circle. The tangent lines intersect the circle at points B and C.



The points B and C divide the circle up into two arcs, which have a ratio of lengths of $3 : 5$. What is the measure of the angle $\angle BAC$?

a) $30°$

b) $45°$

c) $60°$

d) $75°$

e) $90°$

28. If x, y are two distinct positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{37}$$

then what is $x + y$?

a) 345

b) 756

c) 867

d) 1444

e) 2023

29. The largest real solution x to the equation

$$\log_2 x + \log_4 x + \log_8 x = (\log_2 x)(\log_4 x)(\log_8 x)$$

lies in which of the following intervals?

a) $[1, 2)$

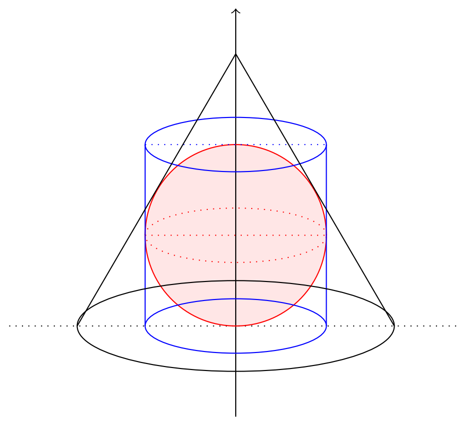
b) $[2, 4)$

c) $[4, 8)$

d) $[8, 16)$

e) $[16, 32)$

30. Consider a cone of revolution with inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of the bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.



the axis of rotation

Find the smallest possible number k for which $V_1 = kV_2$.

a) 1

b) $\frac{4}{3}$

c) $\frac{5}{2}$

d) $\frac{7}{8}$

e) $\frac{9}{5}$