The 50th Annual High School Mathematics Contest

April 17, 2024

Minnesotata State University, Mankato

In memory of Dr. Waters who loved math and the math contest.

Instruction for the Contest

1. Write your **name, grade, school** and **answers** on the answer sheet.

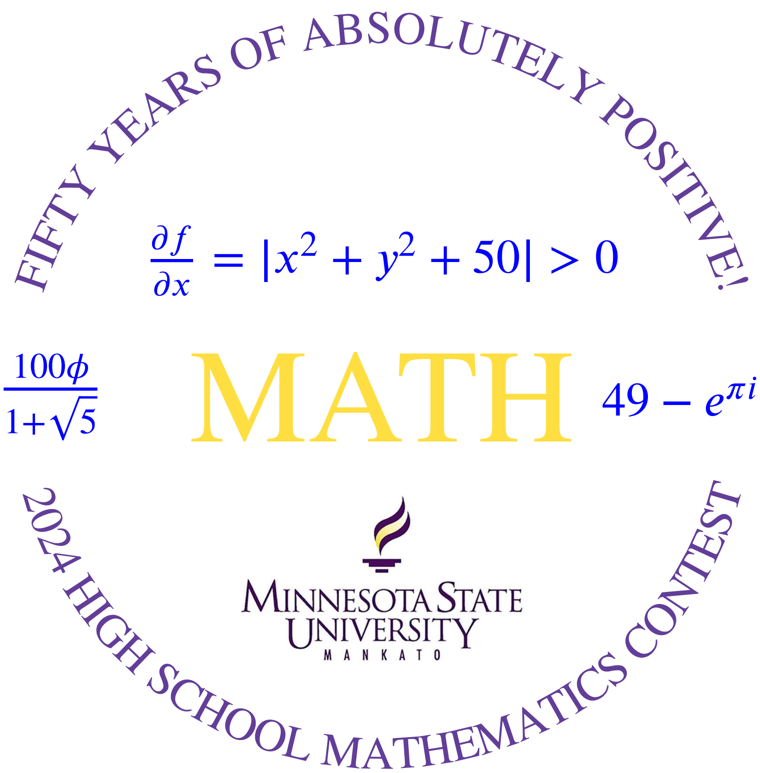
2. You have 90 minutes to work on 30 questions and one tie breaker problem.

3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.

4. You need to write a solution for the tie breaker problem, and the tie breaker will be used only to break any possible ties that arise on the test.

5. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

6. **Submit your answer sheet** and **solution of the tie breaker** at the end of the contest, and keep the exam booklet.



1. If the statement "All problems in this math contest are difficult" is false, then which of the following statements must be true?

I) All problems in this math contest are boring.
II) There is some problem in this math contest that is not difficult.
III) No problem in this math contest is difficult.
IV) Not all problems in this math contest are difficult.

a) II only

b) IV only

c) I and III only

d) II and IV only

e) I, II and IV only

2. Consider the equation

$$x^{9x-8} = x^7.$$

There are two positive numbers that solve the equation. What is the difference between the largest solution and the smallest solution?

a) $\frac{2}{3}$

b) 1

c) $\frac{9}{7}$

d) 3

e) $\frac{72}{7}$

3. The sum of the prime factors of $2^{16} - 1$ is

a) 280

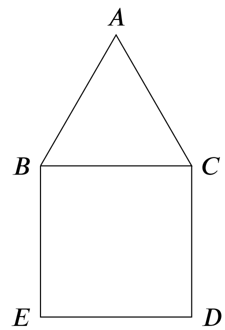
b) 282

c) 273

d) 274

e) 302

4. Suppose that $\triangle ABC$ is an equilateral triangle and that $\square BCDE$ is a square. Find the angle $\angle DAC$ in degrees.



a) 7.5°

b) 10°

c) 15°

d) 20°

e) 30°

5. A sequence a_n is recursively defined by

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} = a_n + 2n \qquad \text{for all } n \geq 1.$$

Find a_{100}.

a) 9899

b) 9901

c) 9903

d) 9905

e) 9909

6. A fair six-sided die is rolled. What is the probability that the product of the five visible numbers is over 150?

a) $\frac{1}{6}$

b) $\frac{2}{6}$

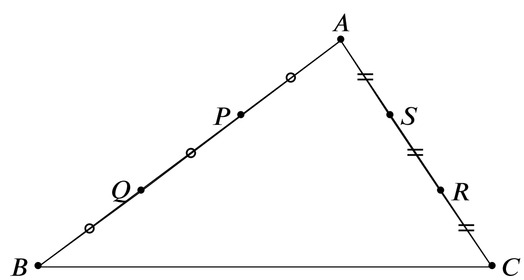
c) $\frac{3}{6}$

d) $\frac{4}{6}$

e) $\frac{5}{6}$

7. On a triangle $\triangle ABC$, two points P and Q are placed on the side AB and another two points R and S are placed on the side AC so that

$$AP = PQ = QB \qquad \text{and} \qquad AS = SR = RC$$



What is the ratio of the area of the quadrilateral $\square PQRS$ to the area of the triangle $\triangle ABC$?

a) $\frac{1}{6}$

b) $\frac{1}{5}$

c) $\frac{1}{4}$

d) $\frac{1}{3}$

e) $\frac{1}{2}$

8. A boss says that the percentage of women in her department is somewhere strictly between 60 and 65 percent. What is the smallest number of employees that could be in the department?

a) 3

b) 5

c) 6

d) 8

e) 10

9. Determine the smallest positive angle x (in radian) for which $-3\sin\left(2x+\frac{\pi}{3}\right)$ takes on its maximum value.

a) $\frac{\pi}{12}$

b) $\frac{\pi}{4}$

c) $\frac{7\pi}{12}$

d) π

e) $\frac{3\pi}{2}$

10. The sequence

$$\log_{12} 162, \quad \log_{12} x, \quad \log_{12} y, \quad \log_{12} z, \quad \log_{12} 1250$$

is an arithmetic progression. What is x?

a) $125\sqrt{3}$

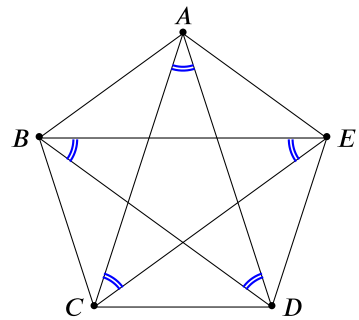
b) 270

c) $162\sqrt{5}$

d) 434

e) $225\sqrt{6}$

11. Inside a regular pentagon ⬠$ABCDE$, a star ★$ACEBDA$ is drawn as shown below.



What is the angle at each vertex of the star?

a) 24°

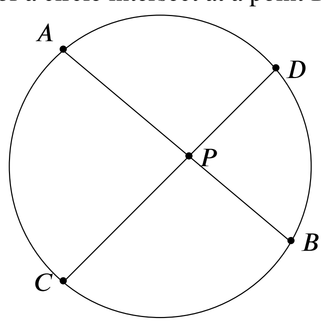
b) 20°

c) 36°

d) 45°

e) 48°

12. The chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at a point P.



If $\angle CPB = 92°$ and the measure of the minor arc $\widehat{AC}$ is 100°, what is the measure of the minor arc $\widehat{BD}$?

a) 38°

b) 47°

c) 68°

d) 76°

e) 108°

13. How many times does the digit "3" appear in the numbers from 1 to 333? (We count with multiplicity. For example, the digit "3" appears in 333 for 3 times.)

a) 99

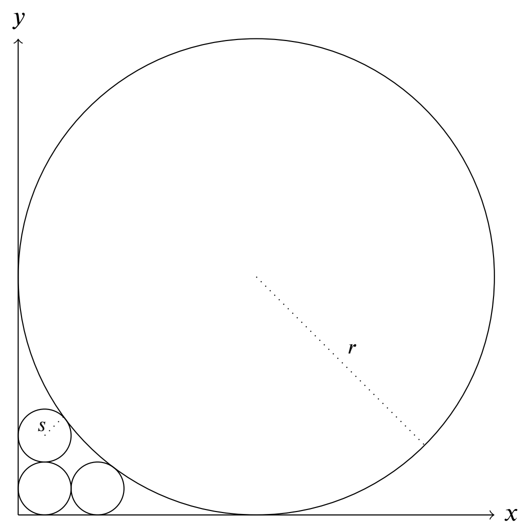
b) 100

c) 101

d) 102

e) 106

14. Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis.



A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?

a) 5

b) 6

c) 9

d) 10

e) 8

15. Evaluate the sum

$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}+\cdots+\frac{1}{\sqrt{119}+\sqrt{121}}$$

a) $\frac{1}{3}$

b) $\frac{1}{\sqrt{3}}$

c) 1

d) 3

e) 5

16. What is the imaginary part of the complex number

$$\left(\cos 12^\circ + i \sin 12^\circ + \cos 48^\circ + i \sin 48^\circ\right)^6 \qquad ?$$

a) $\frac{1}{2}$

b) $\sqrt{3}$

c) 0

d) $\frac{\sqrt{2}+\sqrt{6}}{2}$

e) -2

17. A store had trouble selling an unpopular model of TV. It reduced the price of the TV by a certain percentage in the first week. The second week the TV was still not selling, so it reduced the price again by the same percentage. The third week, it marked down the new price by the same percentage again. At this point, the price of the TV was the same as if the store marked off 65.7% of the original price. What percentage did the store take off each week?

a) 11.4%

b) 20%

c) 21.9%

d) 30%

e) 42.9%

18. For the function $f(x) = \frac{1}{x}$, define

$$f_1(x) = f(x) \quad \text{and} \quad f_{n+1}(x) = f(f_n(x)) \qquad \text{for all } n = 1, 2, 3, ...$$

What is $f_{2024}(x)$?

a) x

b) $\frac{1}{x}$

c) $\frac{x-1}{x}$

d) $\frac{x}{1-x}$

e) $\frac{x}{1-x}$

19. Find the sum of all integers x in $\{1, 2, 3, \cdots, 100\}$ such that 7 divides $x^2 + 15x + 1$.

a) 1458

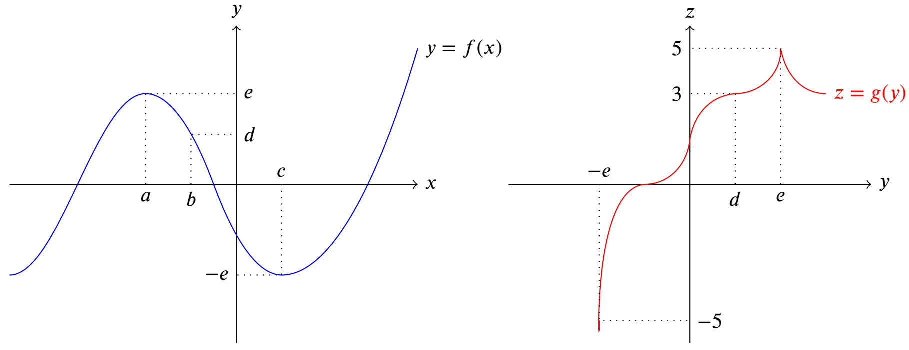
b) 1256

c) 1369

d) 986

e) 2022

20. Consider the graphs of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where a, b, c, d, e are real numbers.



I) $g(f(b)) < 0$

II) $g(f(a)) + g(f(c)) = 0$

III) $g\left(\frac{f(a)+f(b)}{2}\right) \leq g\left(f\left(\frac{a+b}{2}\right)\right)$

Choose all the correct statements.

a) None of them

b) I), II)

c) I), III),

d) II), III)

e) All of them

21. A chess club charges \$23 dollars per person for the annual membership. However, senior citizens only pay \$19. If the club collected a total of \$3,430 in dues, what is the smallest number of senior citizens who could have belonged to the club that year?

a) 0

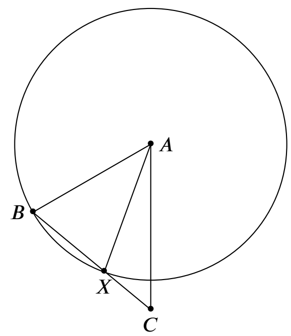
b) 3

c) 5

d) 8

e) 19

22. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius $AB = 86$ intersects $\overline{BC}$ at points B and X. Moreover, $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is the length of $\overline{BC}$?



a) 11

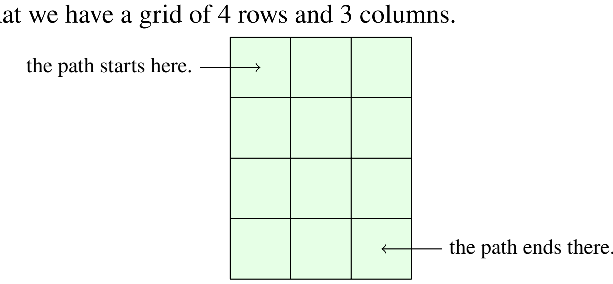
b) 28

c) 33

d) 61

e) 72

23. Suppose that we have a grid of 4 rows and 3 columns.



How many paths are there that start at the upper left corner, end at the lower right corner, and visit each square exactly once?

a) 1

b) 2

c) 3

d) 4

e) 5

24. What is the number of real solutions (x, y, z) to the following system ?

$$\begin{cases} x + yz = 1 \\ y + xz = 1 \\ z + xy = 1 \end{cases}$$

a) 0

b) 1

c) 2

d) more than 3 but finitely many of them

e) infinitely many of them

25. Let a, b, and c be real numbers such that

$$a + b + c = 2 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 12.$$

What is the difference between the maximum and minimum possible values of c?

a) 2

b) $\frac{10}{3}$

c) 4

d) $\frac{16}{3}$

e) $\frac{20}{3}$

26. Let a, b, c be the roots of the equation

$$x^3 + x^2 + 2x + 3 = 0.$$

Find $a^3 + b^3 + c^3$.

a) -4

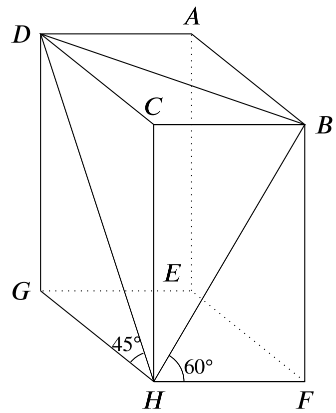
b) -2

c) 0

d) 1

e) 3

27. In the adjoining figure of a rectangular solid, $\angle DHG = 45°$ and $\angle FHB = 60°$. Find the cosine of $\angle BHD$.



a) $\frac{\sqrt{2}}{3}$

b) $\frac{\sqrt{3}}{4}$

c) $\frac{\sqrt{6}}{4}$

d) $\frac{\sqrt{6}-\sqrt{3}}{6}$

e) $\frac{\sqrt{6}-\sqrt{2}}{6}$

28. For certain real numbers a, b and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

a) -9009

b) -8008

c) -7007

d) -6006

e) -5005

29. What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

a) 968

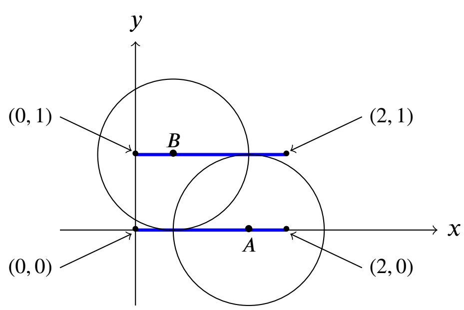
b) 969

c) 970

d) 971

e) 972

30. The center A of a circle radius 1 is chosen uniformly and randomly from the line segment joining $(0,0)$ and $(2,0)$. The center B of another circle of radius 1 is chosen uniformly and randomly from the line segment joining $(0,1)$ to $(2,1)$.



If the centers A, B of two circles are chosen independently, what is the probability that two circles intersect?

a) $\frac{2+\sqrt{2}}{4}$

b) $\frac{3\sqrt{3}+2}{8}$

c) $\frac{2\sqrt{2}-1}{2}$

d) $\frac{2+\sqrt{3}}{4}$

e) $\frac{4\sqrt{3}-3}{4}$

($\star$) **(Tie breaker)** The goal of this problem is to determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive real numbers.

i) Prove that $1 < S$.

ii) Prove that $S < 2$.

iii) Prove that we can make S arbitrarily close to 1.

iv) Prove that we can make S arbitrarily close to 2.

v) Prove that every value of S in the intervale $1 < S < 2$ can be obtained.