The 50th Annual High School Mathematics Contest Solutions

April 17, 2024

Minnesotata State University, Mankato

In memory of Dr. Waters who loved math and the math contest.

Instruction for the Contest

1. Write your **name, grade, school** and **answers** on the answer sheet.

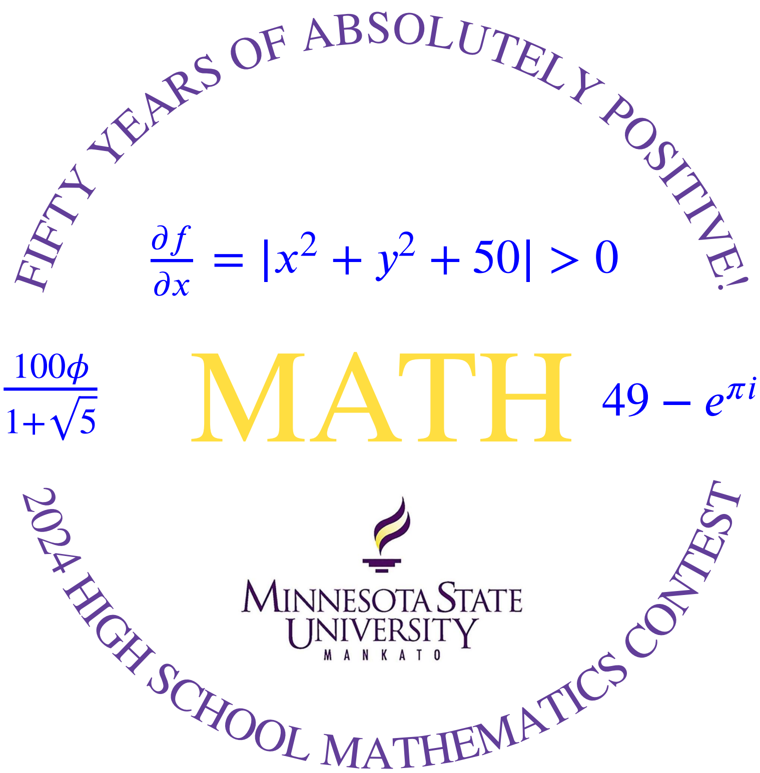
2. You have 90 minutes to work on 30 questions and one tie breaker problem.

3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.

4. You need to write a solution for the tie breaker problem, and the tie breaker will be used only to break any possible ties that arise on the test.

5. The use of any computer, smartphone or calculator during the exam is **NOT** permitted.

6. **Submit your answer sheet** and **solution of the tie breaker** at the end of the contest, and keep the exam booklet.



The Answer Scheet

Question	1	2	3	4	5	6	7	8	9	10
Answer	D	A	B	C	B	D	D	D	C	B

Question	11	12	13	14	15	16	17	18	19	20
Answer	C	D	D	C	E	C	D	A	A	D

Question	21	22	23	24	25	26	27	28	29	30
Answer	C	D	D	D	D	A	C	C	C	E

(Intentionally blank)

1. If the statement "All problems in this math contest are difficult" is false, then which of the following statements must be true?

I) All problems in this math contest are boring.
II) There is some problem in this math contest that is not difficult.
III) No problem in this math contest is difficult.
IV) Not all problems in this math contest are difficult.

a) II only

b) IV only

c) I and III only

d) II and IV only

e) I, II and IV only

Answer: d)

Solution) Since I) is irrelevant to the given information, we cannot determine whether it is true or not. Even if the given statement is false, it does not imply that there are no difficult problems at all in this math contest. Therefore III) is not necessarily true.

2. Consider the equation

$$x^{9x-8} = x^7.$$

There are two positive numbers that solve the equation. What is the difference between the largest solution and the smallest solution?

a) $\frac{2}{3}$

b) 1

c) $\frac{9}{7}$

d) 3

e) $\frac{72}{7}$

Answer: a)

Solution I) Taking the logarithm of both sides

$$x^{9x-8} = x^7 \implies \log x^{9x-8} = \log x^7 \implies (9-8x)\log x = 7\log x$$

we find that

$$(9-8x)\log x = 7\log x \implies \begin{cases} \log x = 1 & \implies x = 1 \\ 9-8x = 7 & \implies x = \frac{5}{3} \end{cases} \implies x_{\max} - x_{\min} = \frac{5}{3} - 1 = \frac{2}{3}.$$

Solution II) This time we work with the exponential function itself. Since

$$x^{9x-8} = x^7 \implies \begin{cases} x = 1 & \text{or} \\ 9x-8 = 7 & \implies x = \frac{5}{3} \end{cases}$$

we find that $x_{\max} - x_{\min} = \frac{5}{3} - 1 = \frac{2}{3}$ is the right anwer.

3. The sum of the prime factors of $2^{16} - 1$ is

a) 280

b) 282

c) 273

d) 274

e) 302

Answer: b)

Solution) Using

$$a^2 - b^2 = (a+b)(a-b)$$

repeatedly, we obtain the factorization of $2^{16} - 1$ into primes

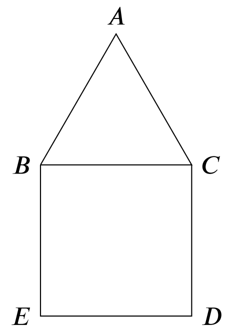
$$\begin{aligned} 2^{16} - 1 &= (2^8 - 1)(2^8 + 1) \\ &= (2^4 - 1)(2^4 + 1)(2^8 + 1) \\ &= (2^2 - 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) \\ &= 3 \cdot 5 \cdot 17 \cdot 257 \end{aligned}$$

and find that

$$3 + 5 + 17 + 257 = 282.$$

Remark) A positive integer n is prime if n is not divisible by the prime numbers less than $\sqrt{n}$. Using this fact, it is easy to check that $3, 5, 17, 257$ are primes.

4. Suppose that $\triangle ABC$ is an equilateral triangle and that $\square BCDE$ is a square. Find the angle $\angle DAC$ in degrees.



a) 7.5°

b) 10°

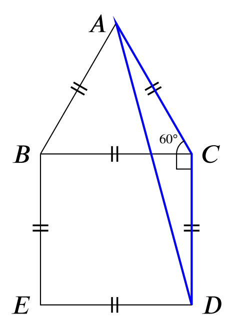
c) 15°

d) 20°

e) 30°

Answer: c)

Solution) Since $AC = CD$, the triangle $\triangle ABC$ is an isosceles and consequently $\angle CAD = \angle CDA$.



Using

$$\angle ACD = \angle ACB + \angle BCD = 60^\circ + 90^\circ = 150^\circ$$

and the angle sum theorem, we find that

$$\angle DAC + \angle CDA + \angle ACD = 180^\circ \implies 2(\angle DAC) + 150^\circ = 180^\circ \implies \angle DAC = 15^\circ.$$

5. A sequence a_n is recursively defined by

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} = a_n + 2n \qquad \text{for all } n \geq 1.$$

Find a_{100}.

a) 9899

b) 9901

c) 9903

d) 9905

e) 9909

Answer: b)

Solution). By the recursive relation,

$$\begin{aligned} a_{100} &= a_{99} + 2 \cdot 99 \\ &= a_{98} + 2 \cdot 98 + 2 \cdot 99 \\ &= a_{98} + 2 \cdot (98 + 99) \\ &= a_{97} + 2 \cdot (97 + 98 + 99) \\ &= \cdots \\ &= a_1 + 2 \cdot (1 + 2 + \cdots + 97 + 98 + 99) \\ &= 1 + 2 \cdot \frac{99(1+99)}{2} = 9901. \end{aligned}$$

6. A fair six-sided die is rolled. What is the probability that the product of the five visible numbers is over 150?

a) $\frac{1}{6}$

b) $\frac{2}{6}$

c) $\frac{3}{6}$

d) $\frac{4}{6}$

e) $\frac{5}{6}$

Answer: d)

Solution) Let k be the invisible number of the rolled die.

i) If $k = 6$, then the product of visible numbers is $\frac{6!}{6} = 120$.
ii) If $k = 5$, then the product of visible numbers is $\frac{6!}{5} = 144$.
iii) If $k = 4$, then the product of visible numbers is $\frac{6!}{4} = 180 > 0$.
iv) If $1 \le k \le 3$, then

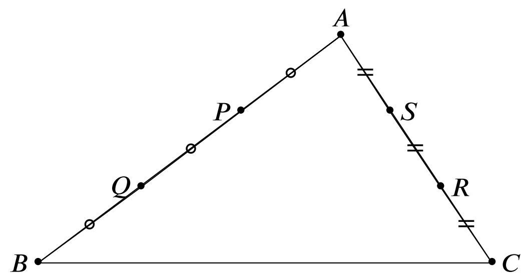
$$(\text{the product of invisible numbers}) = \frac{6!}{k} > \frac{6!}{4} > 150.$$

The desired probability is

$$\frac{\#\{1 \le k \le 4\}}{\#\{1 \le k \le 6\}} = \frac{4}{6}.$$

7. On a triangle $\triangle ABC$, two points P and Q are placed on the side AB and another two points R and S are placed on the side AC so that

$$AP = PQ = QB \qquad \text{and} \qquad AS = SR = RC$$



What is the ratio of the area of the quadrilateral $\square PQRS$ to the area of the triangle $\triangle ABC$?

a) $\frac{1}{6}$

b) $\frac{1}{5}$

c) $\frac{1}{4}$

d) $\frac{1}{3}$

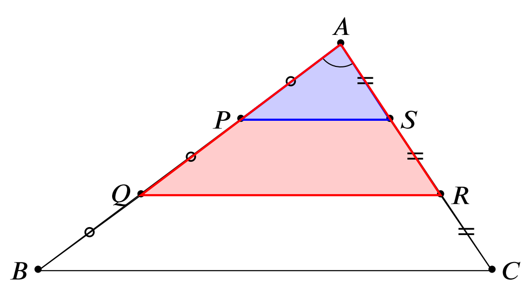
e) $\frac{1}{2}$

Answer: d)

Solution I) We assume that the area of the triangle $\triangle ABC$ is 1. Since $\triangle APS$ and $\triangle ABC$ are similar by SAS (side-angle-side) similarity and

$$AP = \frac{1}{3}AB \qquad \text{and} \qquad AS = \frac{1}{3}AC$$

we find that the area of $\triangle APS$ is $\frac{1}{9}$.

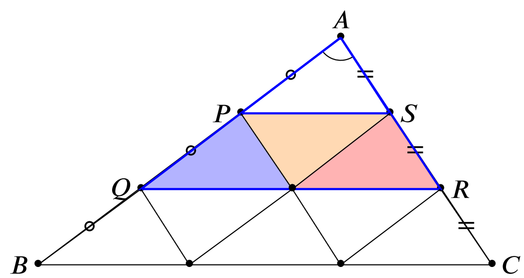


Since $\triangle AQR$ and $\triangle ABC$ are similar by SAS (side-angle-side) similar and

$$AQ = \frac{2}{3}AB \qquad \text{and} \qquad AR = \frac{2}{3}AC$$

we find that the area of $\triangle AQR$ is $\frac{4}{9}$. Therefore, the area of the quadrilateral $\square PQRS$ is $\frac{4}{9} - \frac{1}{9} = \frac{1}{3}$.

Solution II) Since the triangles $\triangle APS$ and $\triangle AQR$ are similar to $\triangle ABC$ by SAS similarity, the sides PS and PQ are parallel to the side BC.



Assume that the area of A is 1. Considering nine smaller triangles that are all congruent to APS, we find that each congruent triangle has area $\frac{1}{9}$ and that the area of the quadrilateral $\square PQRS$ is

$$\frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

Remark) The labelings of points are changed.

8. A boss says that the percentage of women in her department is somewhere strictly between 60 and 65 percent. What is the smallest number of employees that could be in the department?

a) 3

b) 5

c) 6

d) 8

e) 10

Answer: d)

Solution I) Let k be the number of female employees among the n employees. We look for the smallest possible value of n such that $\frac{60}{100} < \frac{k}{n} < \frac{65}{100}$ for some k satisfying $0 < k < n$.

Observing that

$$\frac{60}{100} < \frac{k}{n} < \frac{65}{100} \implies \frac{6n}{10} < k < \frac{13n}{20} \implies \begin{cases} 1.8 < k < 1.95 & \text{if } n = 3 \\ 2.4 < k < 2.6 & \text{if } n = 4 \\ \vdots & \vdots \\ 4.2 < k < 4.55 & \text{if } n = 7 \\ 4.8 < k < 5.2 & \text{if } n = 8 \end{cases}$$

we find that $n = 8$ is the smallest number that gives the right percentage.

Solution II) Let k be the number of female employees among the n employees. This comes down down to considering what possible percentages are possible with fractions of the form $\frac{k}{n}$ for a fixed n. For example, $n = 3$, the possible percentage (rounded to the nearest percentages) are $0\%, 33\%, 67\%$ and 100%, none of which falls into the required range. Continuing in this way with $n = 3, 4, 5, ...$, we find that $n = 8$ is the smallest such a number because $\frac{6}{8} = 0.625$ gives the percentage 62.5% that falls into the required range.

9. Determine the smallest positive angle x (in radian) for which $-3\sin\left(2x + \frac{\pi}{3}\right)$ takes on its maximum value.

a) $\frac{\pi}{12}$

b) $\frac{\pi}{4}$

c) $\frac{7\pi}{12}$

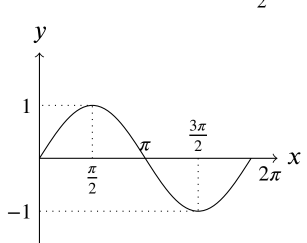
d) π

e) $\frac{3\pi}{2}$

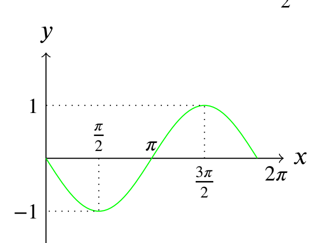
Answer: c)

Solution I) We apply the following series of transformations of $y = \sin x$ to obtain $y = -3\sin\left(2x + \frac{\pi}{3}\right)$ while keeping the maximum value in mind.

i) The transformation $[y = \sin x] \rightarrow [y = -\sin x]$ interchanges the maximum and minimum because the graph is reflected with respect to the x-axis : Since the first minimum of $y = \sin x$ occurs at $x = \frac{3\pi}{2}$, the first maximum of $y = -\sin x$ occurs at $x = \frac{3\pi}{2}$.

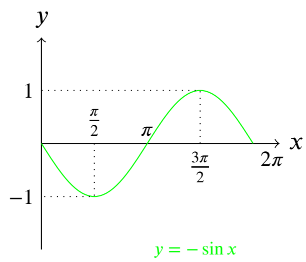


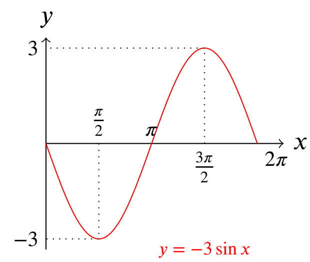
$y = \sin x$



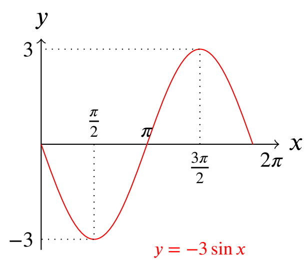
$y = -\sin x$

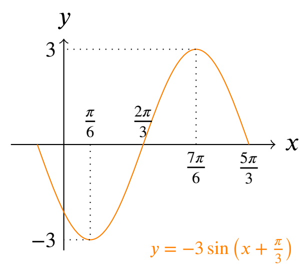
ii) The transformation $[y = -\sin x] \rightarrow [y = -3\sin x]$ scales the graph the vertically by a factor of 3: This transformation does not change where the maximum occurs.





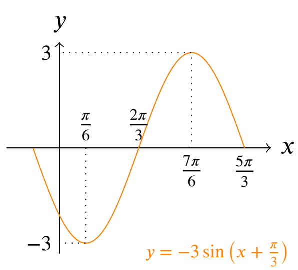
iii) The transfromation $[y = -3\sin x] \rightarrow [y = -3\sin\left(x + \frac{\pi}{3}\right)]$ shifts the graph to the left by $\frac{\pi}{3}$: The maximum of $y = -3\sin\left(x + \frac{\pi}{3}\right)$ occurs at $\frac{3\pi}{2} - \frac{\pi}{3} = \frac{7\pi}{6}$.

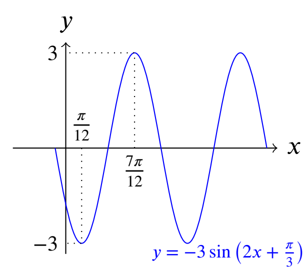




iv) The transformation $[y = -3\sin\left(x + \frac{\pi}{3}\right)] \rightarrow [y = -3\sin\left(2x + \frac{\pi}{3}\right)]$ scales the graph horizontally by a factor of $\frac{1}{2}$: The maximum of $y = -3\sin\left(2x + \frac{\pi}{3}\right)$ occurs at

$$x = \frac{1}{2} \cdot \frac{7\pi}{6} = \frac{7\pi}{12}.$$





Solution II) Since the maximum value of $f(x) = -\sin x$ occurs at $\frac{3\pi}{2}$ over the interval $[0, 2\pi]$, we find that the maximum value of $g(x) = -\sin\left(2x + \frac{\pi}{3}\right)$ occurs at the points x satisfying

$$2x + \frac{\pi}{3} = 2n\pi + \frac{3\pi}{2} \implies x = n\pi + \frac{7\pi}{12} \qquad \text{for integers } n = \cdots, -2, -1, 0, 1, 2, \cdots$$

The smallest positive x value for the maximum of $g(x)$ is $\frac{7\pi}{12}$ when $n = 0$.

10. The sequence

$$\log_{12} 162, \quad \log_{12} x, \quad \log_{12} y, \quad \log_{12} z, \quad \log_{12} 1250$$

is an arithmetic progression. What is x?

a) $125\sqrt{3}$

b) 270

c) $162\sqrt{5}$

d) 434

e) $225\sqrt{6}$

Answer: b)

Solution I) As the sequence

$$\log_{12} 162, \quad \log_{12} x, \quad \log_{12} y, \quad \log_{12} z, \quad \log_{12} 1250$$

is an arithmetic progression, the sequence

$$162, \quad x, \quad y, \quad z, \quad 1250$$

must be a geometric progression. If r is the common ratio of the geometric progression, then

$$1250 = 162r^4 \implies r^4 = \frac{1250}{162} = \frac{2 \cdot 5^4}{2 \cdot 3^4} = \left(\frac{5}{3}\right)^4 \implies r = \frac{5}{3} \qquad (r > 0 \text{ because } x > 0)$$

and so

$$x = 162 \cdot \frac{5}{3} = 270.$$

Solution II) Let d be the common difference of the arithmetic sequence

$$\log_{12} 162, \quad \log_{12} x, \quad \log_{12} y, \quad \log_{12} z, \quad \log_{12} 1250$$

Then

$$\log_{12} 162 + 4d = \log_{12} 1250 \implies 4d = \log_{12} 1250 - \log_{12} 162 = \log \frac{1250}{162}$$
$$\implies d = \log\left(\frac{1250}{162}\right)^{1/4}$$

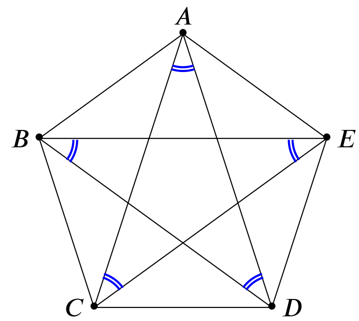
and

$$\log_{12} x = \log_{12} 162 + \log\left(\frac{1250}{162}\right)^{1/4} = \log_{12}\left(162\left(\frac{1250}{162}\right)^{1/4}\right)$$

We find that

$$x = 162\left(\frac{1250}{162}\right)^{1/4} = 162\left(\frac{2 \cdot 5^4}{2 \cdot 3^4}\right)^{1/4} = 162 \cdot \frac{5}{3} = 270.$$

11. Inside a regular pentagon ⬠$ABCDE$, a star ★$ACEBDA$ is drawn as shown below.



What is the angle at each vertex of the star?

a) 24°

b) 20°

c) 36°

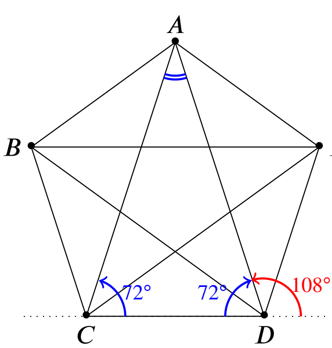
d) 45°

e) 48°

Answer: c)

Solution I) The interior angle at each vertex of any regular pentagon is

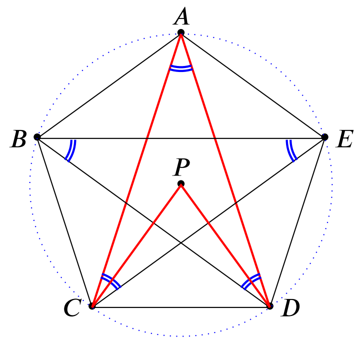
$$\frac{1}{5} \cdot (5-2) \cdot 180^\circ = 108^\circ.$$



As $\overline{AD}$ and $\overline{BC}$ are parallel, the angle $\angle ADC$ of the isosceles $\triangle ACD$ is $180^\circ - 108^\circ = 72^\circ$. By the angle sum of $\triangle ACD$, we find that the common angle is

$$\angle CAD = 180^\circ - 72^\circ - 72^\circ = 36^\circ.$$

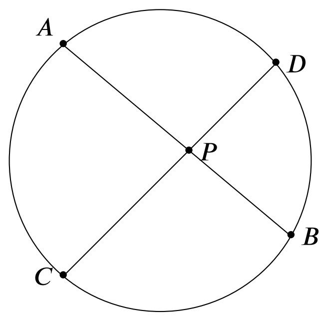
Solution II) Draw a circumscribed circle of the pentagon with center at P.



By the inscribed angle theorem, we find that

$$\angle CAD = \frac{1}{2}\angle CPD = \frac{1}{2} \cdot \left(\frac{1}{5} \cdot 360°\right) = 36°.$$

12. The chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at a point P.



If $\angle CPB = 92°$ and the measure of the minor arc $\widehat{AC}$ is $100°$, what is the measure of the minor arc $\widehat{BD}$?

a) 38°

b) 47°

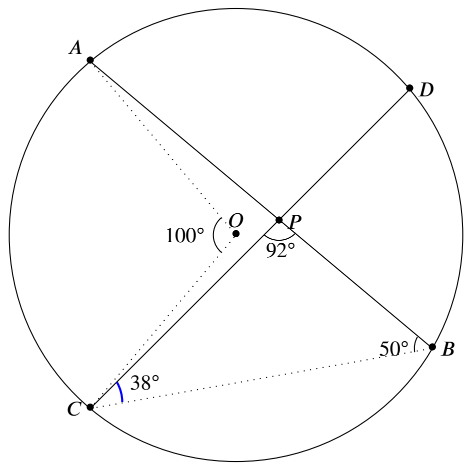
c) 68°

d) 76°

e) 108°

Answer: d)

Solution) Let O be the center of the circle.



By the inscribed angle theorem, we find that

$$\angle ABC = \frac{1}{2}\angle AOC = \frac{1}{2} \cdot 100^\circ = 50^\circ.$$

By the angle sum of the triangle $\triangle PBC$, we find that

$$\angle PCB = 180^\circ - \angle PBC - \angle CPB = 180^\circ - 50^\circ - 92^\circ = 38^\circ.$$

Applying the inscribed angle theorem again, we find

$$\angle DOB = 2 \cdot \angle DCB = 2 \cdot 38^\circ = 76^\circ$$

and so the measurement of the minor arc $\overset{\frown}{AC}$ is 76°.

13. How many times does the digit "3" appear in the numbers from 1 to 333? (We count with multiplicity. For example, the digit "3" appears in 333 for 3 times.)

a) 99

b) 100

c) 101

d) 102

e) 106

Answer: d)

Solution) The digit "3" appears in the ones place in the following numbers

$$3, \quad 13, \quad 23, \quad ,..., \quad 333$$

and there are $(333 - 3)/10 + 1 = 34$ of them. The digit "3" appears in the tens place in the following numbers

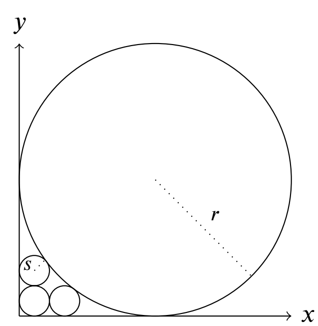
$$30, 31, ..., 39, \quad 130, 131, ..., 139, \quad 230, 231, ..., 239, \quad 330, 331, 332, 333.$$

and there are $10 + 10 + 10 + 4 = 34$ of them. The digit "3" appears in the hundreds place in the following numbers

$$300, 301, 302, ..., 332, 333$$

and there are 34 of them. So the digit "3" appears $34 + 34 + 34 = 102$ times in the numbers from 1 to 333.

14. Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis.



A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?

a) 5

b) 6

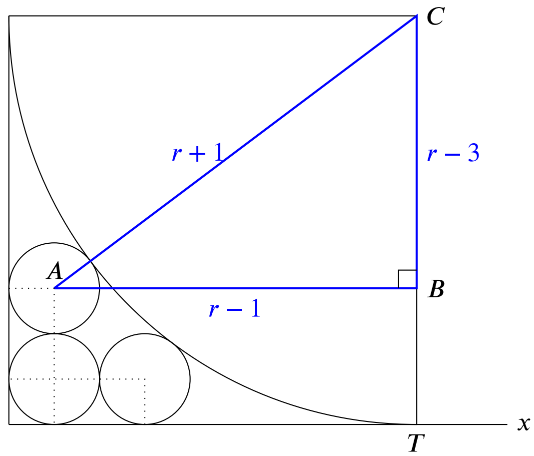
c) 9

d) 10

e) 8

Answer: c)

Solution) Set $s = 1$ so that we only have to find r. Let A, C be the centers of the 3rd circle and the larger circle, respectively and let T be the point at which the larger circle meets the x-axis tangentially. If B is the point on $\overline{CT}$ such that $\overline{AB}$ is perpendicular to $\overline{CT}$, then the triangle $\triangle ABC$ is a right triangle with lengths $AB = r - 1$, $BC = r - 3$ and $AC = r + 1$ as shown below.



Applying the Pythagorean theorem to $\triangle ABC$, we find that

$$(r-3)^2+(r-1)^2=(r+1)^2 \implies r^2-10r+9=0 \implies r=1,9 \implies r=9$$

because $r>1$.

15. Evaluate the sum

$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}+\cdots+\frac{1}{\sqrt{119}+\sqrt{121}}$$

a) $\frac{1}{3}$

b) $\frac{1}{\sqrt{3}}$

c) 1

d) 3

e) 5

Answer: e)

Solution) Rationalizing the denominator

$$\begin{aligned}\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}} &= \frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}\cdot\frac{\sqrt{2k-1}-\sqrt{2k+1}}{\sqrt{2k-1}-\sqrt{2k+1}}\\ &= \frac{\sqrt{2k-1}-\sqrt{2k+1}}{\left(\sqrt{2k-1}\right)^2-\left(\sqrt{2k+1}\right)^2}\\ &= \frac{\sqrt{2k-1}-\sqrt{2k+1}}{-2}=\frac{1}{2}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)\end{aligned}$$

we find that

$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}+\cdots+\frac{1}{\sqrt{121}+\sqrt{119}}$$
$$=\frac{1}{2}(\sqrt{3}-\sqrt{1})+\frac{1}{2}(\sqrt{5}-\sqrt{3})+\frac{1}{2}(\sqrt{7}-\sqrt{5})+\cdots+\frac{1}{2}(\sqrt{119}-\sqrt{117})+\frac{1}{2}(\sqrt{121}-\sqrt{119})$$
$$\stackrel{(*)}{=}\frac{1}{2}(\sqrt{121}-\sqrt{1})$$
$$=\frac{1}{2}(11-1)=5$$

by observing the alternating sum in $(*)$.

16. What is the imaginary part of the complex number

$$\big(\cos 12^\circ + i\sin 12^\circ + \cos 48^\circ + i\sin 48^\circ\big)^6 \qquad ?$$

a) $\frac{1}{2}$

b) $\sqrt{3}$

c) 0

d) $\frac{\sqrt{2}+\sqrt{6}}{2}$

e) -2

Answer: c)

Solution) Grouping real and imaginary terms together and using the sum to product formulas for sine and cosine, we have

$$\begin{aligned}\cos 12^\circ + i\sin 12^\circ + \cos 48^\circ + i\sin 48^\circ &= (\cos 12^\circ + \cos 48^\circ) + i(\sin 12^\circ + i\sin 48^\circ)\\ &= 2\cos\left(\frac{12^\circ+48^\circ}{2}\right)\cos\left(\frac{12^\circ-48^\circ}{2}\right)+2i\sin\left(\frac{12^\circ+48^\circ}{2}\right)\cos\left(\frac{12^\circ-48^\circ}{2}\right)\\ &= 2\cos 18^\circ(\cos 30^\circ + i\sin 30^\circ).\end{aligned}$$

By De Moivre's Theorem, we find that

$$\begin{aligned}(\cos 12^\circ + i\sin 12^\circ + \cos 48^\circ + i\sin 48^\circ)^6 &= (2\cos 18^\circ)^6\big(\cos(6\cdot 30^\circ)+i\sin(6\cdot 30^\circ)\big)\\ &= (2\cos 18^\circ)^6\big(\cos 180^\circ + i\sin 180^\circ\big)\\ &= -(2\cos 18^\circ)^6\end{aligned}$$

is a real number. So its imaginary part is 0.

17. A store had trouble selling an unpopular model of TV. It reduced the price of the TV by a certain percentage in the first week. The second week the TV was still not selling, so it reduced the price again by the same percentage. The third week, it marked down the new price by the same percentage again. At this point, the price of the TV was the same as if the store marked off 65.7% of the original price. What percentage did the store take off

each week?

a) 11.4%

b) 20%

c) 21.9%

d) 30%

e) 42.9%

Answer: d)

Solution) Assume that the original price of the TV is 1 and let r be the markdown each week written in number (not in percentage). The new price is $1 - r$ times the old price for each week passing and so the price of the TV after 3 weeks is $(1 - r)^3$ which must be the same as $1 - \frac{657}{1000} = \frac{343}{1000}$. Solving

$$(1 - r)^3 = \frac{343}{1000} \implies (1 - r)^3 = \frac{7^3}{10^3} \implies 1 - r = \frac{7}{10} \implies r = \frac{3}{10}$$

we find that the discount rate is $\frac{3}{10} = 30\%$.

18. For the function $f(x) = \frac{1}{x}$, define

$$f_1(x) = f(x) \quad \text{and} \quad f_{n+1}(x) = f(f_n(x)) \qquad \text{for all } n = 1, 2, 3, ...$$

What is $f_{2024}(x)$?

a) x

b) $\frac{1}{x}$

c) $\frac{x-1}{x}$

d) $\frac{x}{1-x}$

e) $\frac{x}{1-x}$

Answer: a)

Solution) Using the recursive formula

$$f_{n+1}(x) = f(f_n(x)) \implies (*) \quad f_{n+1}(x) = \frac{1}{f_n(x)}$$

we find that

$$f_2(x) = \frac{1}{f_1(x)} = \frac{1}{\frac{1}{x}} = x \implies f_2(x) = x$$
$$\implies f_3(x) = \frac{1}{x}$$
$$\implies f_4(x) = x$$
$$\implies f_5(x) = \frac{1}{x}$$
$$\vdots$$
$$\implies f_{2024}(x) = x.$$

19. Find the sum of all integers x in $\{1, 2, 3, \cdots, 100\}$ such that 7 divides $x^2 + 15x + 1$.

a) 1458

b) 1256

c) 1369

d) 986

e) 2022

Answer: a)

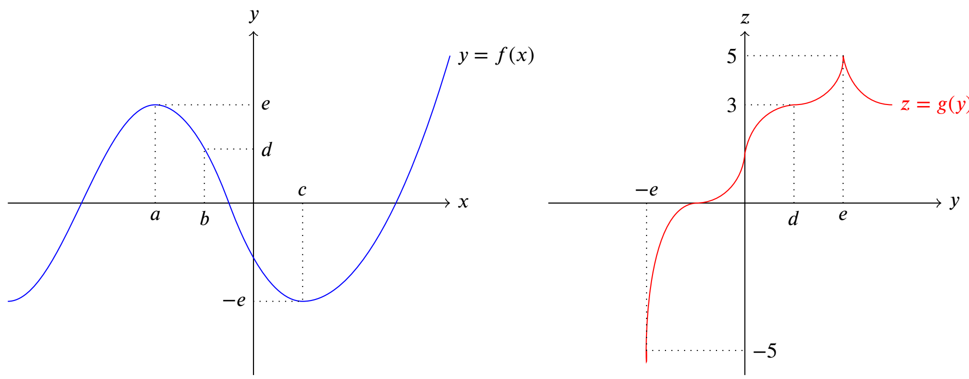
Solution) We look for integer x satisfying

$$\begin{aligned} x^2 + 15x + 1 \equiv 0 \mod 7 &\implies x^2 + x - 6 \equiv 0 \mod 7 \\ &\implies (x-2)(x+3) \equiv 0 \mod 7 \\ &\overset{(*)}{\implies} x \equiv 2 \quad \text{or} \quad x \equiv 4 \mod 7 \end{aligned}$$

In $(*)$, we use the fact that 7 is a prime number. The sum of those integers in $1 \leq x \leq 100$ is

$$\sum_{k=0}^{14}(2+7k) + \sum_{k=0}^{13}(4+7k) = \frac{(2+100)\cdot 15}{2} + \frac{(4+95)\cdot 14}{2} = 1458.$$

20. Consider the graphs of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where a, b, c, d, e are real numbers.



I) $g(f(b)) < 0$

II) $g(f(a)) + g(f(c)) = 0$

III) $g\left(\frac{f(a)+f(b)}{2}\right) \leq g\left(f\left(\frac{a+b}{2}\right)\right)$

Choose all the correct statements.

a) None of them

b) I), II)

c) I), III),

d) II), III)

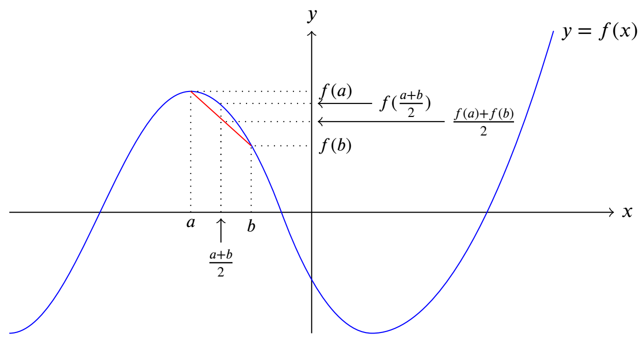
e) All of them

Answer: d)

Solution) i) is false becaue $g(f(b)) = g(d) = 3$. ii) is true because

$$g(f(a)) + g(f(c)) = g(e) + g(-e) = 5 + (-5) = 0.$$

iii) is true: First, we find that $\frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$ by looking at the grah of $y = f(x)$.



Since $g(y)$ is increasing for $d \leq y \leq e$, we find that

$$\frac{f(a)+f(b)}{2} \leq f\Big(\frac{a+b}{2}\Big) \implies g\Big(\frac{f(a)+f(b)}{2}\Big) \leq g\Big(f\big(\frac{a+b}{2}\big)\Big).$$

21. A chess club charges \$23 dollars per person for the annual membership. However, senior citizens only pay \$19. If the club collected a total of \$3,430 in dues, what is the smallest number of senior citizens who could have belonged to the club that year?

a) 0

b) 3

c) 5

d) 8

e) 19

Answer: c)

Solution I) Let x be the number of regular members (without senior discount) and let y be the number of senior members. Then

$$(*) \quad 23x + 19y = 3430 \implies 23x = 3430 - 19y.$$

To solve $(*)$ for nonnegative integers, observe that $3430 - 19y$ must be an integer multiplies of the prime number 23. Considering nonnegative integers y producing an integer solution x in increasing order

$$23x = 3430 - 19y \implies \begin{cases} y = 0 & \implies 23x = 3430 \\ y = 1 & \implies 23x = 3411 \\ y = 2 & \implies 23x = 3392 \\ y = 3 & \implies 23x = 3373 \\ y = 4 & \implies 23x = 3354 \\ y = 5 & \implies 23x = 3335 \implies x = 145 \end{cases}$$

we find that the smallest possible number y of senior citizens is 5.

Solution II) Let x be the number of regular members (without senior discount) and let y be the number of senior members. Then

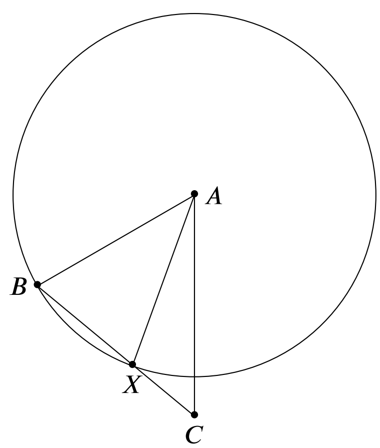
$$(*) \quad 23x + 19y = 3430 \implies 23x = 3430 - 19y.$$

Considering the equation $(*)$ in modulo 23, we find that

$$\begin{aligned} 23x + 19y \equiv 3430 \mod 23 &\implies -4y \equiv 3 \mod 23 \\ &\overset{(\dagger)}{\implies} 24y \equiv -18 \mod 23 \\ &\implies y \equiv 5 \mod 23 \end{aligned}$$

In $(\dagger)$, we multiply both sides by -6. The smallest positive integer solution of $y \equiv 5 \mod 23$ is $y = 5$ and the corresponding smallest number $x = 145$ from $(*)$.

22. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius $AB = 86$ intersects $\overline{BC}$ at points B and X. Moreover, $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is the length of $\overline{BC}$?



a) 11

b) 28

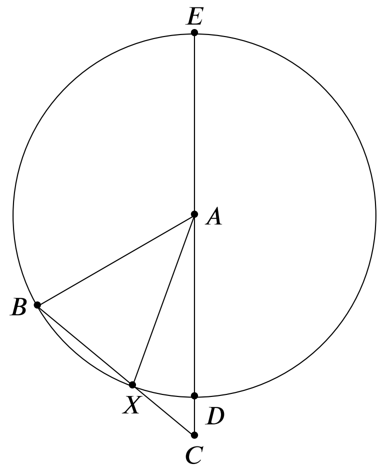
c) 33

d) 61

e) 72

Answer: d)

Solution) Let D and E be the points at which the circle meets the line through A and C.



Let $x = CX$ and $BC = y$, and observe that

$$CD = AC - AD = 97 - 86 = 11 \qquad \text{and} \qquad CE = AC + AE = 97 + 86 = 183 = 3 \cdot 61.$$

Applying the power of a point to the circle with respect to C, we find that

$$CX \cdot CB = CD \cdot CE \implies (*) \quad xy = 3 \cdot 11 \cdot 61.$$

Solving the equation $(*)$ for positive integers x, y with $x < y$, we find that

$$\begin{cases} x = 1 \\ y = 2013 \end{cases} \qquad \begin{cases} x = 3 \\ y = 671 \end{cases} \qquad \begin{cases} x = 11 \\ y = 183 \end{cases} \qquad \text{or} \qquad \begin{cases} x = 33 \\ y = 61 \end{cases}$$

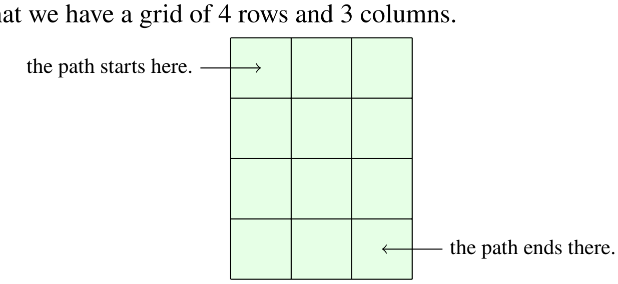
and that

$$y = BC = 61$$

because the triangle inequality applied to $\triangle ACX$ implies

$$CX + AX > AC \implies x + 86 > 97 \implies x > 11.$$

23. Suppose that we have a grid of 4 rows and 3 columns.



How many paths are there that start at the upper left corner, end at the lower right corner, and visit each square exactly once?

a) 1

b) 2

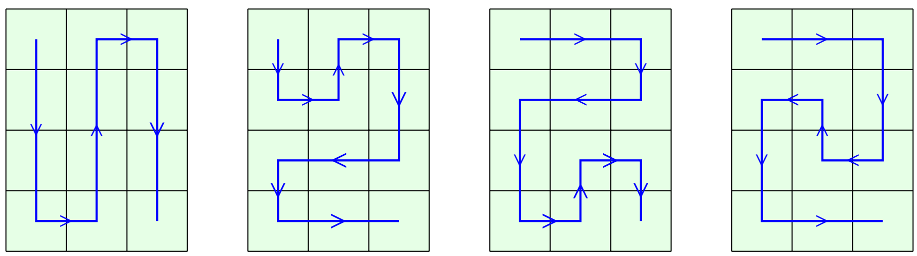
c) 3

d) 4

e) 5

Answer: d)

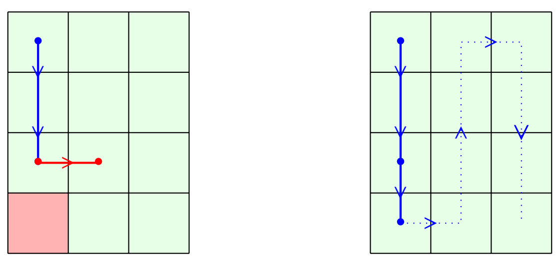
Solution) We find that there are 4 paths satisfying the requirement.



path I path II path IV-a path IV-b

On the 1st move, we can go either down or right. This gives us 4 possible cases:

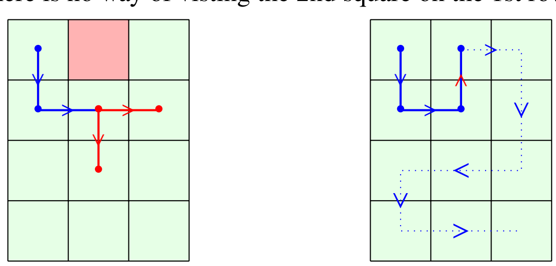
I) First, we go **down and then down**. If we move to the right from the current position, then there is no way of visiting the lower left corner.



bad path good path

This forces us to move down from the current posisiton. This kind of analysis gives the unique path I in this case.

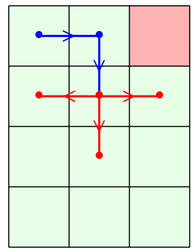
II) First, we go **down and then right**. If we move down or to the right from the current position, then there is no way of visting the 2nd square on the 1st row.



bad path good path

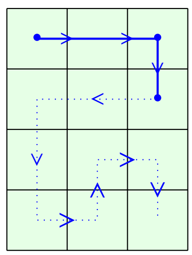
This forces us to move up from the current posisiton. This kind of analysis gives the unique path II in this case.

III) First, we go **right and then down**. If we move to the left, to the right or down from the current position, then there is no way of visiting the upper right corner square. We are at a dead end.

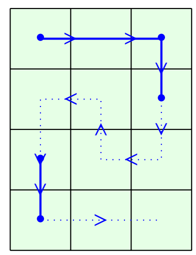


bad path

IV) After we go **right and then right**, we must move down from the current position. There are two possible path since then. All the remaining moves are forced.



good path



good path

24. What is the number of real solutions (x, y, z) to the following system ?

$$\begin{cases} x + yz = 1 \\ y + xz = 1 \\ z + xy = 1 \end{cases}$$

a) 0

b) 1

c) 2

d) more than 3 but finitely many of them

e) infinitely many of them

Answer: d)

Solution) Subtracting 2nd equation from the 1st equation, we find that

$$(x + yz) - (y + xz) = 0 \implies (x - y) + (y - x)z = 0 \implies (x - y)(1 - z) = 0$$

and so $x = y$ or $z = 1$.

If $x = y$, then the system becomes (after eliminating y)

$$\begin{cases} x + xz = 1 \\ x + xz = 1 \\ z + x^2 = 1 \end{cases} \implies \begin{cases} x + xz = 1 \\ z + x^2 = 1 \end{cases} \implies \begin{cases} x + x(1 - x^2) = 1 \\ z = 1 - x^2 \end{cases}$$

and the equation $x + x(1 - x^2) = 1$ can be solved:

$$x^3 - 2x + 1 = 0 \implies (x - 1)(x^2 + x - 1) = 0 \implies x = 1 \text{ or } \alpha \text{ or } \beta$$

where $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$ are the roots of $x^2 + x - 1 = 0$. Since $y = x$ and $z = 1 - x^2$ are uniquely determined for each value of x, we find that there are 3 distinct solutions in this case, i.e.,

$$(x, y, z) = (1, 1, 0), \quad (\alpha, \alpha, 1 - \alpha^2), \quad (\beta, \beta, 1 - \beta^2).$$

If $z = 1$, then the system becomes

$$\begin{cases} x + y = 1 \\ y + x = 1 \\ 1 + xy = 1 \end{cases} \implies \begin{cases} x + y = 1 \\ xy = 0 \end{cases} \implies \begin{cases} x = 0 \\ y = 1 \end{cases} \quad \text{or} \quad \begin{cases} x = 0 \\ y = 1 \end{cases}$$

we find that there are two solutions in this case, i.e.,

$$(x, y, z) = (0, 1, 1), \quad (1, 0, 1).$$

There are at least 5 solutions. If we work with 1st and 3rd equation or 2nd and 3rd equation, we may obtain more solutions but it is clear that there are finitely many of them by symmetry on x, y, z.

Remark) Since α and β are roots of $x^2 + x - 1 = 0$, we find that $1 - \alpha^2 = \alpha$ and $1 - \beta^2 = \beta$. We can write 5 solutions above as

$$(x, y, z) = (1, 1, 0), \quad (0, 1, 1) \quad (1, 0, 1), \quad (\alpha, \alpha, \alpha) \quad (\beta, \beta, \beta).$$

Since the system is symmetric on x, y, z and the 5 solutions are symmetric on x, y, z, we find that the system has exactly 5 solutions.

25. Let a, b, and c be real numbers such that

$$a + b + c = 2 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 12.$$

What is the difference between the maximum and minimum possible values of c?

a) 2

b) $\frac{10}{3}$

c) 4

d) $\frac{16}{3}$

e) $\frac{20}{3}$

Answer: d)

Solution) We have

$$(*) \qquad \begin{cases} a+b = 2-c \\ a^2+b^2 = 12-c^2 \end{cases}$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned}(1^2+1^2)(a^2+b^2) \geq (a+b)^2 &\implies 2(a^2+b^2) \geq (a+b)^2 \\ &\implies 2(12-c^2) \geq (2-c)^2 \\ &\implies 3c^2-4c-20 \leq 0 \\ &\implies (c+2)(3c-10) \leq 0 \\ &\implies -2 \leq c \leq \frac{10}{3}.\end{aligned}$$

The difference between the maximum and minimum possible values of c is $\frac{10}{3}-(-2) = \frac{16}{3}$.

Remark) The Cauchy-Schwarz inequality has equality when $a = b$. Solving $(*)$ with $a = b$, we find that the minimum value $c = -2$ actually occurs at $(a, b) = (2, 2)$ and the maximum value $c = \frac{10}{3}$ occurs at $(a, b) = \left(-\frac{2}{3}, -\frac{2}{3}\right)$.

26. Let a, b, c be the roots of the equation

$$x^3 + x^2 + 2x + 3 = 0.$$

Find $a^3 + b^3 + c^3$.

a) -4

b) -2

c) 0

d) 1

e) 3

Answer: a)

Solution I) By Vieta's formula, we obtain

$$a+b+c = -1, \qquad ab+bc+ca = 2 \qquad \text{and} \qquad abc = -3$$

which imply

$$\begin{aligned}a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= (-1)^2 - 2\cdot 2 = -3.\end{aligned}$$

Using the factorization formula

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

we find that

$$\begin{aligned} a^3+b^3+c^3 &= (a^3+b^3+c^3-3abc)+3abc \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc \\ &= (-1)\cdot(-3-2)+3\cdot(-3) \\ &= -4 \end{aligned}$$

Solution II) By Vieta's formula, we obtain

$$a+b+c=-1, \qquad ab+bc+ca=2 \qquad \text{and} \qquad abc=-3$$

which imply

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2-2(ab+bc+ca) \\ &= (-1)^2-2\cdot 2=-3. \end{aligned}$$

Observing that

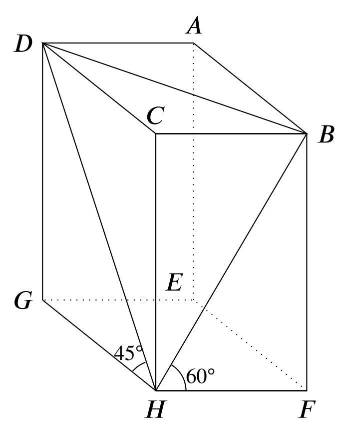
$$\begin{aligned} (a+b+c)^3 &= a^3+b^3+c^3+3(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)+6abc \\ &= a^3+b^3+c^3+3\left[(a^2+b^2+c^2)(a+b+c)-a^3-b^3-c^3)\right]+6abc \end{aligned}$$

we find that

$$\begin{aligned} 2(a^3+b^3+c^3) &= -(a+b+c)^3+3(a^2+b^2+c^2)(a+b+c)+6abc \\ &= -(-1)^3+3\cdot(-3)\cdot(-1)+6\cdot(-3) \\ &= -8 \end{aligned}$$

and so $a^3+b^3+c^3=-4$.

27. In the adjoining figure of a rectangular solid, $\angle DHG = 45°$ and $\angle FHB = 60°$. Find the cosine of $\angle BHD$.



a) $\frac{\sqrt{2}}{3}$

b) $\frac{\sqrt{3}}{4}$

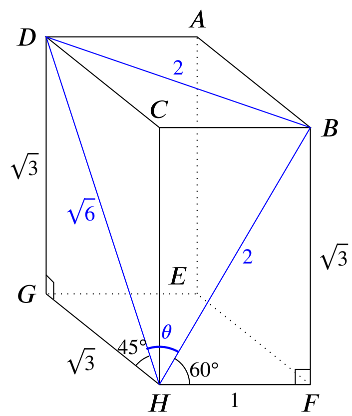
c) $\frac{\sqrt{6}}{4}$

d) $\frac{\sqrt{6}-\sqrt{3}}{6}$

e) $\frac{\sqrt{6}-\sqrt{2}}{6}$

Answer: c)

Solution) Let $\theta = \angle BHD$ and let $HF = 1$ without loss of generality.



Since $\triangle BFH$ is a right triangle with 30°-90°-60° angles, we find that $BF = \sqrt{3}$ and consequently $DG = \sqrt{3}$. Since $\triangle DGH$ is a right triangle with 45°-90°-45° angles, we find that $GF = \sqrt{3}$ and $DH = \sqrt{6}$. Applying the law of cosine to the triangle $\triangle BHD$, we find that

$$\cos\theta = \frac{2^2 + (\sqrt{6})^2 - 2^2}{2 \cdot 2 \cdot \sqrt{6}} = \frac{\sqrt{6}}{4}.$$

28. For certain real numbers a, b and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

a) -9009

b) -8008

c) -7007

d) -6006

e) -5005

Answer: c)

Solution) To eliminate the x^4 term, we consider

$$h(x) = f(x) - xg(x) \implies h(x) = (1-a)x^3 + (b-1)x^2 + 90x + c.$$

Let α, β, γ be distinct roots of $g(x) = 0$. Then α, β, γ are also the roots of $h(x)$ because $f(x)$ and $g(x)$ share the roots α, β, γ. Since both $g(x)$ and $h(x)$ have three distinct roots α, β, γ and they have the same degree (of degree 3), we must have $h(x) = (1-a)g(x)$ or equivalently

$$(1-a)x^3 + (b-1)x^2 + 90x + c = (1-a)x^3 + a(1-a)x^2 + (1-a)x + 10(1-a)$$

which provides a system of equations for a, b, c

$$\begin{cases} a(1-a) = b - 1 \\ 1 - a = 90 \\ 10(1-a) = c \end{cases} \implies \begin{cases} a = -89 \\ b = -8009 \\ c = 900 \end{cases}$$

Therefore, we find that

$$f(1) = 1 + 1 + b + 100 + c = 1 + 1 + (-8009) + 100 + 900 = -7007.$$

29. What is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$?

a) 968

b) 969

c) 970

d) 971

e) 972

Answer: c)

Solution) Thinking of possible cancellations, we may work with the "symmetric" number

$$\left(\sqrt{3} + \sqrt{2}\right)^6 + \left(\sqrt{3} - \sqrt{2}\right)^6.$$

By the binomial theorem, we find that

$$\begin{aligned}\left(\sqrt{3} + \sqrt{2}\right)^6 + \left(\sqrt{3} - \sqrt{2}\right)^6 &= \sum_{k=0}^{6} \binom{6}{k} \left(\sqrt{3}\right)^{6-k} \left(\sqrt{2}\right)^k + \sum_{k=0}^{6} \binom{6}{k} \left(\sqrt{3}\right)^{6-k} \left(-\sqrt{2}\right)^k \\ &= 2\left[\binom{6}{0}\left(\sqrt{3}\right)^6 + \binom{6}{2}\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^2 + \binom{6}{4}\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^4 + \binom{6}{6}\left(\sqrt{2}\right)^6\right] \\ &= 2\left[3^3 + 15 \cdot 3^2 \cdot 2 + 15 \cdot 3 \cdot 2^2 + 2^3\right] = 970.\end{aligned}$$

Since

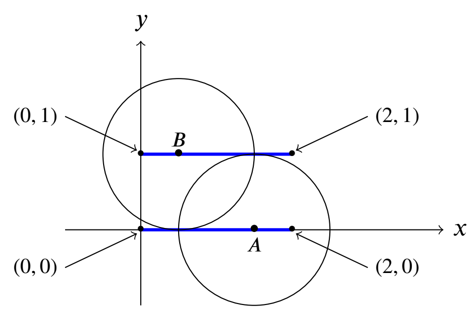
$$0 < \sqrt{3} - \sqrt{2} = \frac{1}{\sqrt{3} + \sqrt{2}} < 1 \implies 0 < \left(\sqrt{3} - \sqrt{2}\right)^6 < 1$$

and

$$\left(\sqrt{3} + \sqrt{2}\right)^6 < \underbrace{\left(\sqrt{3} + \sqrt{2}\right)^6 + \left(\sqrt{3} - \sqrt{2}\right)^6}_{970} < \left(\sqrt{3} + \sqrt{2}\right)^6 + 1$$

we find that the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$ is 970.

30. The center A of a circle radius 1 is chosen uniformly and randomly from the line segment joining $(0, 0)$ and $(2, 0)$. The center B of another circle of radius 1 is chosen uniformly and randomly from the line segment joining $(0, 1)$ to $(2, 1)$.



If the centers A, B of two circles are chosen independently, what is the probability that two circles intersect?

a) $\frac{2+\sqrt{2}}{4}$

b) $\frac{3\sqrt{3}+2}{8}$

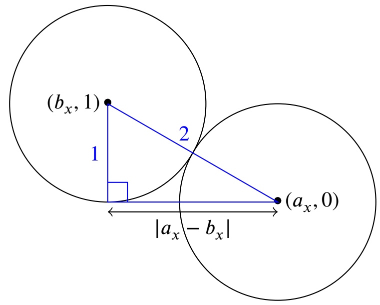
c) $\frac{2\sqrt{2}-1}{2}$

d) $\frac{2+\sqrt{3}}{4}$

e) $\frac{4\sqrt{3}-3}{4}$

Answer: e)

Solution) Let $A = (a_x, 0)$ and $B = (b_x, 1)$. Two circles meet tangentially when $|a_x - b_x| = \sqrt{3}$ by the Pythagorean theorem as shown in the figure below.



Now it is clear that

$$\text{two circles} \begin{cases} \text{intersect} & \text{if } |a_x - b_x| < \sqrt{3} \\ \text{meet tangentially} & \text{if } |a_x - b_x| = \sqrt{3} \\ \text{do not intersect} & \text{if } |a_x - b_x| > \sqrt{3} \end{cases}$$

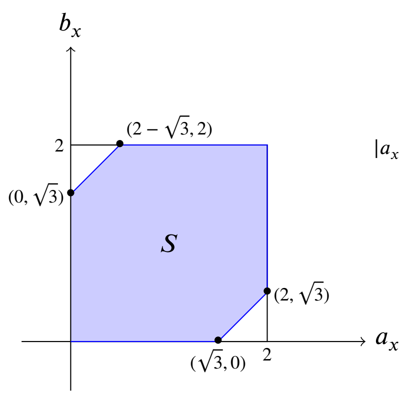
The probability of intersecting circles is $\frac{\text{area}(S)}{\text{area}(T)}$ where

$$S = \{(a_x, b_x) : |a_x - b_x| \le \sqrt{3} \quad \text{and} \quad 0 \le a_x \le 2, 0 \le b_x \le 2\}$$

and

$$T = \{(a_x, b_x) : 0 \le a_x \le 2 \quad \text{and} \quad 0 \le b_x \le 2\}.$$

To compute the areas, we sketch the regions S and T in the plane



$$|a_x - b_x| \le \sqrt{3} \implies a_x - \sqrt{3} \le b_x \le a_x + \sqrt{3}$$

The area of T is 4 and the area of S is

$$\begin{aligned} \text{area}(S) &= 4 - (\text{the sum of areas of the corners}) \\ &= 4 - (2 - \sqrt{3})^2 \\ &= 4\sqrt{3} - 3. \end{aligned}$$

The desired probability is

$$\frac{\text{area}(S)}{\text{area}(T)} = \frac{4\sqrt{3} - 3}{4}.$$

($\star$) **(Tie breaker)** The goal of this problem is to determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive real numbers.

i) Prove that $1 < S$.

ii) Prove that $S < 2$.

iii) Prove that we can make S arbitrarily close to 1.

iv) Prove that we can make S arbitrarily close to 2.

v) Prove that every value of S in the intervale $1 < S < 2$ can be obtained.

Solution) i) Observe that

$$\begin{aligned} S &= \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \\ &> \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1. \end{aligned}$$

ii) Since S is symmetric on a, b, c, d, we may assume that $a \le b \le c \le d$ and so

$$a + b + d \ge a + b + c \quad \text{and} \quad b + c + d \ge a + b + c.$$

We obtain that

$$\begin{aligned} S &= \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \\ &\le \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} + \frac{d}{a+c+d} \\ &\le 1 + \frac{d}{a+c+d} < 1 + 1 = 2. \end{aligned}$$

Another way of showing $S < 2$ is to observe that

$$\begin{aligned} S &= \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \\ &< \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{c+d} + \frac{d}{c+d} \\ &= 1 + 1 = 2. \end{aligned}$$

iii) We write

$$f(a, b, c, d) = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

and use the symbol $p \to q$ to mean that the number p gets arbitrarily closer to q. Then

$$f(1, \epsilon, \epsilon^2, \epsilon^2) = \frac{1}{1+\epsilon+\epsilon^2} + \frac{\epsilon}{1+\epsilon+\epsilon^2} + \frac{\epsilon^2}{\epsilon+\epsilon^2+\epsilon^2} + \frac{\epsilon^2}{1+\epsilon^2+\epsilon^2} \to 1+0+0+0 = 1$$

as $\epsilon \to 0$.

iv) Observe that

$$f(1, \epsilon, 1, \epsilon) = \frac{1}{1+\epsilon+\epsilon} + \frac{\epsilon}{1+\epsilon+1} + \frac{1}{\epsilon+1+\epsilon} + \frac{\epsilon}{1+1+\epsilon} \to 1+0+1+0 = 2$$

as $\epsilon \to 0$.

v) Since the function $f(a, b, c, d)$ varies continuously as (a, b, c, d) varies on the connected region $\{(a, b, c, d) : a, b, c, d > 0\}$ and

$$f(1, \epsilon, \epsilon^2, \epsilon^2) \to 1 \qquad \text{and} \qquad f(1, \epsilon, 1, \epsilon) \to 2 \qquad \text{as} \quad \epsilon \to 0$$

we find that $S = f(a, b, c, d)$ can take all the values in the interval $1 < S < 2$.