The 49th Annual High School Mathematics Contest Solutions
April 19, 2023
Minnesotata State University, Mankato
In memory of Dr. Waters who loved math and the math contest.

## Instruction for the Contest

1. Write your answer on the answer sheet.
2. You have 90 minutes to work on 30 questions and one tie breaker problem.
3. This is a multiple choice test (except for the tie breaker problem) and there is no penalty for a wrong answer.
4. You need to write a solution for the tie breaker problem on the answer sheet, and the tie breaker will be used only to break any possible ties that arise on the test.
5. The use of any computer, smartphone or calculator during the exam is NOT permitted.
6. Submit your answer sheet at the end of the contest and keep the exam booklet.


2

1. Write $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ with a rational denominator. The result is
a) $\frac{3+2 \sqrt{6}+\sqrt{60}}{12}$
b) $\frac{3+2 \sqrt{6}+\sqrt{15}}{6}$
c) $\frac{4+2 \sqrt{6}+\sqrt{60}}{12}$
d) $\frac{3+\sqrt{6}+\sqrt{15}}{6}$
e) $\frac{3+\sqrt{6}+\sqrt{12}}{6}$

## Answer: d)

Solution) To rationalize the denominator, we multiply the fraction by $\sqrt{2}+\sqrt{3}+\sqrt{5}$ first and then by $\sqrt{6}$ as follows:

$$
\begin{aligned}
\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} & =\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}} \cdot \frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} \\
& =\frac{2+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^{2}-5} \\
& =\frac{2+\sqrt{6}+\sqrt{10}}{2 \sqrt{6}} \\
& =\frac{2+\sqrt{6}+\sqrt{10}}{2 \sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\
& =\frac{2 \sqrt{6}+6+\sqrt{60}}{12} \\
& =\frac{3+\sqrt{6}+\sqrt{15}}{6}
\end{aligned}
$$

2. In Springfield where the following people live

a heated debate developed:
Homer: Bart, the probability that you win in a math competition is 50\% because either you win or you lose. Therefore, your chance of winning
is $50 \%+50 \%=100 \%$ if you participate $\mathrm{AMC}^{1}$ and $\mathrm{MMC}^{2}$ math contests.
Participate these contests and win!
Marge: Homie, you are so smart.
Bart: Cool! Dad, I will win one of the contests.
Lisa: Dad! You are wrong.

Mr. Burns: All of you are wrong!
Who is right?
a) Homer is right.
b) Marge is right.
c) Bart is right.
d) Lisa is right.
e) Mr. Burns is right.

## Answer: d)

Solution) Homer has a wrong idea on probability. The probability that Bart wins a competition is $50 \%$ has no logical ground, and so Lisa is right.
3. A ship sails 10 miles in a straight line from $A$ to $B$, turns through an angle between $45^{\circ}$ and $60^{\circ}$, and then sails another 20 miles to $C$. Let $A C$ be measured in miles.


Which of the following intervals contains $A C^{2}$ ?
a) $[400,500)$

[^0]4
b) $[500,600)$
c) $[600,700)$
d) $[700,800)$
e) $[800,900)$

## Answer: d)

Solution) Let $\theta$ be supplement to the turning angle of the ship. Then $120^{\circ}<\theta<135^{\circ}$.


Using the law of cosines to the triangle $\triangle A B C$, we find that

$$
A C^{2}=10^{2}+20^{2}-2 \cdot 10 \cdot 20 \cdot \cos \theta=500-400 \cos \theta
$$

Since $\cos \left(135^{\circ}\right)<\cos \theta<\cos \left(120^{\circ}\right)$, we find that

$$
700=500-400 \cos 120^{\circ}<A C^{2}<500-400 \cos 135^{\circ}=500+200 \sqrt{2}
$$

because $\cos 120^{\circ}=-1 / 2$ and $\cos 135^{\circ}=-\sqrt{2} / 2$. Noting that

$$
500+200 \sqrt{2} \approx 500+200 \cdot 1.4 \approx 780
$$

we find that

$$
700<A C^{2}<800
$$

4. The function $f(x)=2 x+\frac{12}{x}$ is defined on the positive real numbers $x>0$. At which value of $x$ does $f(x)=2 x+\frac{12}{x}$ assume its minimum value?
a) $x=1$
b) $x=\sqrt{2}$
c) $x=\sqrt{6}$
d) $x=\sqrt{12}$
e) $x=6$

Answer: c)

Solution) Using the arithmetic mean-geometric mean inequality: for all $a, b>0$,

$$
a+b \geq 2 \sqrt{a b} \quad \text { with equality if and only if } a=b
$$

we find that the minimum of $f(x)$ occurs

$$
f(x)=2 x+\frac{12}{x} \geq 2 \sqrt{2 x \cdot \frac{12}{x}}=4 \sqrt{6}
$$

if and only if $x$ satisfies

$$
2 x=\frac{12}{x} \Longrightarrow x^{2}=6 \Longrightarrow x=\sqrt{6}
$$

5. Let $r, s, t$ be the roots of $x^{3}-6 x^{2}+5 x-7=0$. Find

$$
\frac{1}{r^{2}}+\frac{1}{s^{2}}+\frac{1}{t^{2}}
$$

a) $-\frac{59}{49}$
b) $-\frac{39}{29}$
c) $\frac{25}{49}$
d) $-\frac{16}{25}$
e) 3

## Answer: a)

Solution I) Using Vita's Theorem, we find that

$$
\begin{cases}r+s+t & =6 \\ r s+s t+t r & =5 \\ r s t & =7\end{cases}
$$

and

$$
\begin{aligned}
\frac{1}{r^{2}}+\frac{1}{s^{2}}+\frac{1}{t^{2}} & =\left(\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\right)^{2}-2\left(\frac{1}{r} \cdot \frac{1}{s}+\frac{1}{s} \cdot \frac{1}{t}+\frac{1}{t} \cdot \frac{1}{s}\right) \\
& =\left(\frac{r s+s t+t r}{r s t}\right)^{2}-2\left(\frac{r+s+t}{r s t}\right) \\
& =\left(\frac{5}{7}\right)^{2}-2\left(\frac{6}{7}\right) \\
& =-\frac{59}{49}
\end{aligned}
$$

Solution II) The polynomial whose roots are the reciprocals of the roots of the original
polynomial can be obtained by reversing the order of coefficients of the original polynomial because

$$
\left(\frac{1}{x}\right)^{3}-6\left(\frac{1}{x}\right)^{2}+5\left(\frac{1}{x}\right)-7=0 \Longrightarrow(*) \quad-7 x^{3}+5 x^{2}-6 x+1=0
$$

Since $\frac{1}{r}, \frac{1}{s}, \frac{1}{t}$ are roots of the polynomial equation $(*)$, we find that

$$
\begin{aligned}
\frac{1}{r^{2}}+\frac{1}{s^{2}}+\frac{1}{t^{2}} & =\left(\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\right)^{2}-2\left(\frac{1}{r} \cdot \frac{1}{s}+\frac{1}{s} \cdot \frac{1}{t}+\frac{1}{t} \cdot \frac{1}{s}\right) \\
& =\left(-\frac{5}{-7}\right)^{2}-2\left(\frac{-6}{-7}\right) \\
& =-\frac{59}{49}
\end{aligned}
$$

by applying Vieta's formula to $(*)$.
6 . If $2^{16^{x}}=16^{2^{x}}$, then $x$ is equal to
a) 2
b) $\frac{3}{4}$
c) $\frac{2}{3}$
d) $\frac{1}{3}$
e) 4

Answer: c)
Solution) To express both sides of the equation in base 2 we write the left hand side as

$$
16^{2 x}=\left(2^{4}\right)^{2^{x}}=2^{4 \cdot 2^{x}}
$$

and observe that

$$
\begin{aligned}
2^{16^{x}}=16^{2^{x}} & \Longrightarrow 2^{16^{x}}=2^{4 \cdot 2^{x}} \\
& \xlongequal[(*)]{\Longrightarrow} 16^{x}=4 \cdot 2^{x} \\
& \Longrightarrow 8^{x} \cdot 2^{x}=4 \cdot 2^{x} \\
& \Longrightarrow 8^{x}=4 \\
& \Longrightarrow x=\frac{2}{3} .
\end{aligned}
$$

Remark) In $(*)$ above, we use the fact that $f(x)=2^{x}$ is a one-to-one function:

$$
2^{a}=2^{b} \Longrightarrow a=b
$$

7. At the end of a shift, a store clerk notices that the average value of all the coins left in the till is 20 cents. Furthermore if there was one more quarter, the average value of all the coins would be 21 cents. How many pennies are in the till?
(The coins are all pennies, worth 1 cent, nickles, worth 5 cents, dimes, worth 10 cents, and quarters, worth 25 cents.)
a) 0
b) 1
c) 3
d) 5
e) 7

## Answer: a)

Solution) Let $v$ be the total value of the coins in the till and $n$ the number of coins in the till. Then we have $v / n=20$. The fact that another quarter would make the average value 21 implies:

$$
\frac{v+25}{n+1}=21 \Longrightarrow v+25=21(n+1) \Longrightarrow v-21 n=-4
$$

This is equivalent to the system of linear equations:

$$
\left\{\begin{array}{l}
v-20 n=0 \\
v-21 n=-4
\end{array}\right.
$$

which can be solved by $v=80, n=4$.
Next we can simply reason based on cases depending on the number of quarters. If there are two or fewer quarters the maximum value of four coins is 70 cents, and four quarters are worth 100 cents. So there must be three quarters, implying the last coin is a nickle. Hence there are no pennies.
8. In $\triangle A B C, A B=13, A C=5$, and $B C=12$. Points $M$ and $N$ lie on $A C$ and $B C$, respectively, with $C M=C N=4$. Points $J$ and $K$ are on $A B$ so that $M J$ and $N K$ are perpendicular to $A B$. What is the area of pentagon $C M J K N$ ?

(The right triangle $A B C$ has area $1 / 2 \cdot A C \cdot B C=1 / 2 \cdot 5 \cdot 12=30$ ).
a) 15
b) $\frac{81}{5}$
c) $\frac{205}{12}$
d) $\frac{240}{13}$
e) 20

## Answer: d)

Solution) We are given the following figure:


Observe that the right triangles $\triangle A M J$ and $\triangle A B C$ are similar and that $A M=A C-$ $C M=5-4=1$. Since the ratio of the hypotenuse of $\triangle A M J$ to that of $\triangle A B C$ is $A M: A B=1: 13$, the ratio of the area of $\triangle A M J$ to that of $\triangle A B C$ is $1^{2}: 13^{2}$ and so

$$
(\text { the area of } \triangle A M J)=\left(\frac{1}{13}\right)^{2} \cdot 30 .
$$

Also, observe that the right triangles $\triangle N B K$ and $\triangle A B C$ are similar and that $B N=$ $B C-C N=12-4=8$. Since the ratio of the hypotenuse of $\triangle N B K$ to that of $\triangle A B C$ is $B N: A B=8: 13$, the ratio of the area of $\triangle N B K$ to that of $\triangle A B C$ is $8^{2}: 13^{2}$ and so

$$
\text { (the area of } \triangle N B K)=\left(\frac{8}{13}\right)^{2} \cdot 30 .
$$

We find that
(the area of $\triangle C M J K N)=($ the area of $\triangle A B C)-($ the area of $\triangle A M J)-($ the area of $\triangle N B K)$

$$
\begin{aligned}
& =30-\left(\frac{1}{13}\right)^{2} \cdot 30-\left(\frac{8}{13}\right)^{2} \cdot 30 \\
& =\frac{240}{13}
\end{aligned}
$$

9. Suppose that we have two cubes. The first cube has volume 8 , and the second has twice the surface area of the first. What is the length of the sides of the second cube?
a) 2
b) $2 \sqrt{2}$
c) 4
d) $3 \sqrt{2}$
e) $3 \sqrt[3]{4}$

## Answer: b)

Solution I) The first cube has sides of length $\sqrt[3]{8}=2$, so its surface area is $6 \cdot 2^{2}=24$. Therefore the second cube has surface area 96 , so its sides have length:

$$
\sqrt{\frac{48}{6}}=\sqrt{8}=2 \sqrt{2}
$$

Solution II) Another solution is based on the fact that the area of a square is proportional to the square of the side length. Therefore to double the area we must increase the side length by a factor of $\sqrt{2}$. As the first cube must have sides of length 2 , the second cube has sides of length $2 \sqrt{2}$.
10. Find the smallest positive integer $n$ such that

$$
\left\{\begin{array}{l}
\text { the remainder is } 2 \text { if } n \text { is divided by } 3 \\
\text { the remainder is } 3 \text { if } n \text { is divided by } 5 \text { and } \\
\text { the remainder is } 2 \text { if } n \text { is divided by } 7 .
\end{array}\right.
$$

a) 5
b) 18
c) 23
d) 128
e) 233

Answer: c)
Solution I) The positive integers $n$ leaving the remainder 2 when $n$ is divided by 3 are

$$
2, \quad 5, \quad 8, \quad 11, \quad 14, \quad 17, \quad 20, \quad 23, \quad 26, \ldots .
$$

The positive integers $n$ leaving the remainder 3 when $n$ is divided by 5 are
$3, \quad 8,13,18,23,28,33,38,43, \ldots$.
The positive integers $n$ leaving the remainder 2 when $n$ is divided by 7 are
$2, ~ 9, ~ 16, ~ 23, ~ 30, ~ 37, ~ 44, ~ 51, ~ 58, ~ \ldots .$.

We find that the smallest such an integer is $n=23$.
Solution II) We look for integers of the form

$$
n=a \cdot 5 \cdot 7+b \cdot 3 \cdot 7+c \cdot 3 \cdot 5
$$

where integers $a, b, c$ satisfying

$$
\left\{\begin{array}{l}
n \equiv 2 \bmod 3 \Longrightarrow 2 a \equiv 2 \bmod 3 \Longrightarrow a \equiv 1 \bmod 3 \\
n \equiv 3 \bmod 5 \Longrightarrow b \equiv 3 \bmod 5 \Longrightarrow b \equiv 3 \bmod 5 \\
n \equiv 2 \bmod 7 \Longrightarrow c \equiv 2 \bmod 7 \Longrightarrow c \equiv 2 \bmod 7
\end{array}\right.
$$

Considering the congruence modulo $105=3 \cdot 5 \cdot 7$, we find that

$$
n=1 \cdot 5 \cdot 7+3 \cdot 3 \cdot 7+2 \cdot 3 \cdot 5 \equiv 35+63+30 \equiv 128 \equiv 23 \bmod 105
$$

11. If $f\left(\frac{2 x+1}{3 x+2}\right)=x$ for all $x$ except $-\frac{2}{3}$, determine $f(x)$.
a) $f(x)=\frac{3 x+2}{2 x+1}$
b) $f(x)=\frac{2 x-1}{2-3 x}$
c) $f(x)=\frac{3 x-2}{2 x-1}$
d) $f(x)=\frac{3 x-2}{2 x+1}$
e) $f(x)=\frac{2 x-1}{3 x-2}$.

## Answer: b)

Solution) Setting $y=\frac{2 x+1}{3 x+2}$, we solve for $x$ :

$$
\begin{aligned}
y=\frac{2 x+1}{3 x+2} & \Longrightarrow y(3 x+2)=2 x+1 \\
& \Longrightarrow(2-3 y) x=2 y-1 \\
& \Longrightarrow x=\frac{2 y-1}{2-3 y}
\end{aligned}
$$

Changing the name of the variable, we find that

$$
f(y)=\frac{2 y-1}{2-3 y} \Longrightarrow f(x)=\frac{2 x-1}{2-3 x} .
$$

12. The ratio of the measures of two acute angles is $5: 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the measures of the two angles?
a) $75^{\circ}$
b) $90^{\circ}$
c) $135^{\circ}$
d) $150^{\circ}$
e) $270^{\circ}$

## Answer: c)

Solution) Let $\theta$ and $\phi$ be the angle measures, where $90^{\circ}>\theta>\phi>0$. Then we know:

$$
\frac{\theta}{\phi}=\frac{5}{4} \quad \text { and } \quad \frac{90^{\circ}-\phi}{90^{\circ}-\theta}=2
$$

which implies

$$
\theta=\frac{5}{4} \phi \quad \text { and } \quad 2 \theta-\phi=90^{\circ} \Longrightarrow \phi=60^{\circ} \quad \text { and } \quad \theta=75^{\circ}
$$

Therefore, the sum of two angles is $135^{\circ}$.
13. The figure is constructed from 11 line segments, each of which has length 2 . The area of pentagon $A B C D E$ can be written as $\sqrt{m}+\sqrt{n}$, where $m$ and $n$ are positive integers.


What is $m+n$ ?
a) 23
b) 24
c) 22
d) 20
e) 21

## Answer: a)

Solution) Label two points inside the pentagon as $G$ and $F$ as shown in the figure below and let $M$ be the midpoint of $C D$.


Draw diagonals $A C$ and $A D$ to split the pentagon into three parts. The triangle $\triangle A B C$ is an isosceles triangle with $\angle A B C=120^{\circ}$ because $\triangle A B F$ and $\triangle C F B$ are equilateral triangles.


From the triangle $\triangle A B C$ with internal angles $30^{\circ}-120^{\circ}-30^{\circ}$, we find that $A C=2 \sqrt{3}$ and that

$$
(\text { the area of } \triangle \mathrm{ABC})=\frac{1}{2} \cdot 2 \cdot 2 \sin \left(120^{\circ}\right)=\sqrt{3}
$$

Likewise, the area of $\triangle A E D$ is also $\sqrt{3}$. To find the height of $A M$, we apply the Pythagorean theorem to the right triangle $\triangle A C M$

$$
A M^{2}=A C^{2}-C M^{2}=(2 \sqrt{3})^{2}-1^{2}=11 \Longrightarrow A M=\sqrt{11}
$$

and find that

$$
(\text { the area of } \triangle \mathrm{ACD})=\frac{1}{2} \cdot 2 \cdot \sqrt{11}=\sqrt{11}
$$

The area of the pentagon is

$$
\begin{aligned}
(\text { area of } \triangle A B C D E) & =(\text { area of } \triangle A B C)+(\text { area of } \triangle A C D)+(\text { area of } \triangle A E D) \\
& =\sqrt{3}+\sqrt{11}+\sqrt{3} \\
& =\sqrt{11}+2 \sqrt{3}=\sqrt{11}+\sqrt{12}
\end{aligned}
$$

and the sum $m+n=11+12=23$.
14. A store has a pile of cans arranged so that the top row has 3 cans and each subsequent row has two more cans than the one above it.


There are 120 cans in the whole pile. How many rows are in the pile?
a) 10
b) 11
c) 12
d) 13
e) 14

## Answer: a)

Solution) One way to solve this is to simply add up odd numbers starting from 3 until we get 120 :

$$
\begin{aligned}
& \underbrace{3+5}_{2 \text { rows }}=8 \\
& \underbrace{3+5+7}_{3 \text { rows }}=15 \\
& \vdots \\
& \underbrace{3+5+7+9+11+13+15+17+19+21}_{10 \text { rows }}=120
\end{aligned}
$$

We get to 120 at the 10th term.
Alternatively, we use the fact that the sum of the first $k$ odd numbers is $k^{2}$ :

$$
\underbrace{1+3+5+\cdots+(2 k-1)}_{k \text { terms }}=k^{2} .
$$

Since $120=121-1=11^{2}-1$, we conclude that if the pile had started with 1 can above the three cans in the top row, then there would be eleven rows. But since that row is not actually there, there are ten rows.
15. In the expansion of $\left(x y-2 y^{-3}\right)^{16}$, find the power of $x$ in the term that does not contain $y$.
a) 12
b) 8
c) 6
d) 7
e) 10

## Answer: a)

Solution) Using the binomial expansion

$$
\begin{aligned}
\left(x y-2 y^{-3}\right)^{16} & =\sum_{k=0}^{16}\binom{16}{k}(x y)^{k} \cdot\left(-2 y^{-3}\right)^{16-k} \\
& =\sum_{k=0}^{16}\binom{16}{k}(-2)^{16-k} \cdot x^{k} \cdot y^{k-3(16-k)}
\end{aligned}
$$

we find that the $k$ in the term not containing $y$ must satisfy

$$
k-3(16-k)=0 \Longrightarrow k=12
$$

and that the power of $x$ in this term is 12 .
16. The last digit of $7^{7^{7}}$ is
a) 7
b) 1
c) 3
d) 9
e) 5

## Answer: c)

Solution) We shall find $7^{7^{7}} \bmod 10$. Since

$$
7^{2} \equiv-1 \quad \text { and } \quad 7^{4} \equiv 1 \quad \bmod 10
$$

we consider the remainder of the power $7^{7} \bmod 4$. Because

$$
7^{2}=49 \equiv 1 \quad \bmod 4 \Longrightarrow 7^{7}=\left(7^{2}\right)^{3} \cdot 7 \equiv 1 \cdot 3 \equiv 3 \bmod 4
$$

we can write $7^{7}=4 n+3$ for some integer $n$ and find that

$$
7^{7^{7}} \equiv 7^{4 n+3} \equiv\left(7^{4}\right)^{n} \cdot 7^{3} \equiv 1 \cdot(-3)^{3}=-27 \equiv 3 \quad \bmod 10
$$

The last digit of $7^{7^{7}}$ in the decimal representation is 3 .
17. Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per
hour. At the beginning of the trip, $a b c$ miles was displayed on the odometer, where $a b c$ is a 3-digit number with $a \geq 1$ and $a+b+c \leq 7$. At the end of the trip, the odometer showed $c b a$ miles. What is $a^{2}+b^{2}+c^{2}$ ?
a) 26
b) 27
c) 36
d) 37
e) 41

## Answer: d)

Solution I) Let the number of hours Danica drove be $k$. Then we know that

$$
100 a+10 b+c+55 k=100 c+10 b+a \Longrightarrow(*) \quad 9 c-9 a=5 k .
$$

We know from (*) that $k$ is divisible by 9 , i.e., $k=9,18,27, \ldots$. If $k \geq 18$ then

$$
a b c+55 k=100 a+10 b+c+55 \cdot 18 \geq 100 \cdot 1+55 \cdot 18 \geq 1090
$$

which makes $a b c+55 k$ a 4-digit number (impossible). We find that $k$ must be 9 and therefore $c-a=5$. Because $a+b+c \leq 7$ and $a \geq 1$, we find that

$$
a=1, c=6, \text { and } b=0 \Longrightarrow a^{2}+b^{2}+c^{2}=6^{2}+0^{2}+1^{2}=37 .
$$

Solution II) We know that the number of miles she drove is divisible by 5, so $a$ and $c$ must either be the equal or differ by 5 . We can quickly conclude that the former is impossible, because then $a b c$ and $c b a$ are the same. So $a$ and $c$ must be 5 apart. Because we know that $c>a$ and $a+c \leq 7$ and $a \geq 1$, we find that the only possible values for $a$ and $c$ are 1 and 6 , respectively. Because $a+b+c \leq 7, b=0$. Therefore, we have

$$
a=1, c=6, \text { and } b=0 \Longrightarrow a^{2}+b^{2}+c^{2}=6^{2}+0^{2}+1^{2}=37 .
$$

18. If circular arcs $\overparen{A C}$ and $\overparen{B C}$ have centers at $B$ and $A$, respectively, then there exists a circle tangent to both $\overparen{A C}$ and $\overparen{B C}$, and to $\overline{A B}$.


If the length of the arc $\overparen{B C}$ is $4 \pi$, then the area of the circle is
a) $\frac{36 \pi}{2}$
b) $\frac{81 \pi}{4}$
c) $\frac{49 \pi}{5}$
d) $\frac{67 \pi}{6}$
e) $\frac{2023 \pi}{49}$

## Answer: b)

Solution) Note the triangle $A B C$ is equilateral because $A B, A C, A C$ is the common radius, and so $\angle C A B=60^{\circ}$.


Considering the circular sector with arc $\widehat{B C}$ centered at $A$ of radius $A B$, we find that

$$
\frac{1}{6}(2 \pi) A B=4 \pi \Longrightarrow A B=12
$$

Draw a perpendicular from the center $O$ of the circle to the side $A B$, and call this length $r$, and call the foot $M$. Draw an extended line from $A$ to $O$ which meets the circular arc $B C$ tangentially at $M^{\prime}$ and

$$
12=A M^{\prime}=O A+O M^{\prime} \text { and } O M^{\prime}=r \Longrightarrow O A=12-r .
$$

Applying the Pythagorean Theorem to $\triangle A M O$, we find that

$$
r^{2}+6^{2}=(12-r)^{2} \Longrightarrow r=\frac{9}{2}
$$

Finally, we see that the area of the circles is $\pi \cdot\left(\frac{9}{2}\right)^{2}=\frac{81 \pi}{4}$.
19. Suppose that we choose an integer from 1 to 400 (inclusive), where we have equal probability of selecting any integer in the range. What is the probability that we select an even integer which has a 3 as one of its digits?
a) $53 / 400$
b) $3 / 20$
c) $13 / 80$
d) $33 / 200$
e) $3 / 16$

## Answer: c)

Solution) We have 400 numbers in this range. If an integer has a 3 in the ones place, it will never be even and so this case is eliminated.

$$
\square \square \boxed{3} \Longrightarrow \text { eliminated! }
$$

So we only need to count the even integers with a 3 in the tens or hundreds places.
First we count even integers $n$ having a 3 in the tens place. Such an integer $n$ will have to have 0,1 or 2 in the hundreds place, 3 in the tens place, and $0,2,4,6,8$ in the ones place

and so it is in the range $1 \leq n \leq 299$. Furthermore, every choice of one of those numbers in the hundreds, tens and ones places will produce an even integer with a 3 as one of its digits. Therefore there are $3 \cdot 1 \cdot 5=15$ integers of this type. (It is also possible to simply list them all.)

Next, we count even integers $n$ having a 3 in the hundreds place. Such an integer $n$ is in the range $300 \leq n \leq 399$.


All of these integers will have a 3 as a digit, so we only need to worry about how many of them are even. There are 50 such integers. One way to show this is the fact that there are 100 integers in this range with half of them being even. Alternatively, we can form an integer by selecting 3 in the hundreds place, anything from 0 through 9 in the tens place, and $0,2,4,6,8$ in the ones place.

Since 400 does not have a 3 , we conclude that there are $50+15=65$ numbers of this type. Therefore the probability of selecting one of them will be $65 / 400=13 / 80$.
20. Let $n$ be an integer. If the tens digit of $n^{2}$ is 7 , what is the last digit of $n^{2}$ ?
a) 4
b) 1
c) 6
d) 9
e) 5

## Answer: c)

Solution) Let $n=10 x+y$, where $x, y$ are integers such that $1 \leq x \leq 9$ and $0 \leq y \leq 9$. Since

$$
n^{2}=100 x^{2}+20 x y+y^{2}
$$

we find that the tens digit of $y^{2}$ must be odd because the tens digit of $20 x y$ is even and the tens digit of $n$ is odd. For the tens digit of $y^{2}$ to be odd, $y$ must be 4 or 6 . In both cases, the last digit of $y^{2}$ is 6 and the last digit of $n^{2}$ is 6 .
21. Suppose we expand the following product:

$$
\left(1+x+\cdots+x^{50}\right)\left(1+x+\cdots+x^{25}\right)^{2}
$$

What will be the coefficient in front of $x^{10}$ ?
a) 48
b) 55
c) 60
d) 66
e) 72

## Answer: d)

Solution) There are a variety of ways to approach this problem, though ultimately we will need to determine how many ways there are to choose one power of $x$ from each of the three parts of the problem so that their combined powers add to 10 .
$\left(1+x+\cdots+x^{a}+\cdots+x^{50}\right)\left(1+x+\cdots+x^{b}+\cdots+x^{25}\right)\left(1+x+\cdots+x^{c}+\cdots+x^{25}\right)=\left(\sum_{a+b+c=10} 1\right) x^{a+b+c}+\cdots$
This can be rephrased as a question about how many nonnegative integer solutions there are to $a+b+c=10$, where $a, b$ and $c$ are the power of $x$ we take from each of the three terms. Since $10 \leq 25$ we don't have to worry about setting a maximum on the solution. This problem type has a known solution of

$$
\binom{10+3-1}{3-1}=\binom{12}{2}=66
$$

Alternatively, we can break into cases. Generally if we pick $x^{n}$ from the first part, where $0 \leq n \leq 10$, then we will have to pick two terms from the remaining parts whose powers add to $10-n$. There will be $10-n+1$ ways to do this, i.e. by selecting $0,1, \ldots, 10-n$, so the answer is:

$$
1+2+3+4+5+6+7+8+9+10+11=66
$$

22. Suppose that Alice and Bob play the following game: They put a certain number of beads in a pile, which varies from game to game. They then take turns, starting with Alice. On each turn a player can take one or two beads. The player who takes the last bead wins. Both Alice and Bob are experts at their game and always make the best possible move. Which is the only possible true statement?
a) Alice won the games that started with 2,8 and 22 beads.
b) Bob won the games that started with 3,7 and 12 beads.
c) Alice won the games that started with 1,6 and 10 beads.
d) Bob won the games that started with 2, 5 and 16 beads.
e) Alice won the games that started with 1, 2 and 3 beads.

## Answer: a)

Solution) By examining what happens for smaller piles of beads, a pattern can be found about who will win. If there are 1 or 2 beads, then Alice can win immediately. If there are 3 beads, then no matter what Alice does she will need to leave 1 or 2 beads immediately, so Bob will win. (This reasoning actually eliminates answers d) and e) immediately, making it possible to guess.)

Continuing this reasoning, a player will lose if all possible moves lead to a situation where the other player can win. This will happen every multiple of 3 , so Alice will win if the number of beads is not a multiple of 3 and Bob will win if the number of beads is a multiple of 3 . The only statement consistent with this fact is a).
23. Consider the graph of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ where $a, b, c, d$ are real numbers.

i) $f(f(a))=b$
ii) $f(a)+f(b)+f(c)>0$
iii) $f(c)>c$
iv) $f\left(\frac{b+c}{2}\right)>\frac{f(b)+f(c)}{2}$

Choose all the correct statements.
a) None of them
b) ii), iii)
c) i), iii),iv)
d) ii), iii), iv)
e) All of them

## Answer: d)

Solution) We look at the graph of $f$.

i) is false because $f(f(a))=f(b)=-b$.
ii) is true because $f(a)+f(b)+f(c)=b+(-b)+d=d>0$.
iii) is true because $f(c)=d>c$ where $c$ is a negative number and $d$ is a positive number.
iv) is true because the average $\frac{f(b)+f(c)}{2}$ of $f(b)$ and $f(c)$ is below the value $f\left(\frac{b+c}{2}\right)$ of the average $\frac{b+c}{2}$ of $b$ and $c$ on the $y$-axis.
24. Let $f$ be a function defined by the following properties:

- $f(1)=1$
- For any positive integer $n$ we have $f(2 n)=n \cdot f(n)$.

What is the value of $f\left(2^{50}\right)$ ?
a) $2^{50}$
b) $2^{125}$
c) $2^{624}$
d) $2^{1225}$
e) $2^{4950}$

## Answer: d)

Solution) Using the recurrence relation $f(2 n)=n \cdot f(n)$, we find that

$$
\begin{aligned}
f\left(2^{50}\right)=f\left(2 \cdot 2^{49}\right) & =2^{49} \cdot f\left(2^{49}\right) \\
& =2^{49} \cdot f\left(2 \cdot 2^{48}\right) \\
& =2^{49} \cdot 2^{48} \cdot f\left(2^{48}\right) \\
& =\cdots \\
& =2^{49} \cdot 2^{48} \cdots \cdots \cdot 2^{2} \cdot 2^{1} \cdot f(1) \\
& =2^{49} \cdot 2^{48} \cdots \cdots \cdot 2^{2} \cdot 2^{1} \cdot 1 \\
& =2^{49+\cdots+48+\cdots+2+1} \\
& =2^{1225}
\end{aligned}
$$

In the sum on the exponent,

$$
1+2+\cdots+48+49=\frac{49 \cdot 50}{2}=1225
$$

was used. Aternatively, we can get a similar result by expanding "upwards," i.e. by calculating:

$$
\begin{aligned}
& f(1)=1 \Longrightarrow f(2)=2 \cdot f(1)=2^{1} \\
& \Longrightarrow f\left(2^{2}\right)=2 \cdot f(2)=2^{2} \cdot 2^{1} \\
& \Longrightarrow f\left(2^{3}\right)=2 \cdot f\left(2^{2}\right)=2^{3} \cdot 2^{2} \cdot 2^{1} \\
& \vdots \\
& \Longrightarrow f\left(2^{50}\right)=2^{49} \cdot 2^{48} \cdots \cdot 2^{2} \cdot 2^{1}=2^{49+\cdots+48+\cdots+2+1}=2^{1225}
\end{aligned}
$$

25. A cylinderical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet.


What is the volume of water, in cubic feet?
a) $24 \pi-36 \sqrt{2}$
b) $24 \pi-24 \sqrt{3}$
c) $36 \pi-36 \sqrt{3}$
d) $36 \pi-24 \sqrt{2}$
e) $48 \pi-36 \sqrt{3}$

## Answer: e)

Solution) Any vertical cross-section of the tank is the disk of radius 4 with center $O$ and the shaded region indicates water filled. We label points $A, B, C$ as shown below.


The line segment $O C$ meets $A B$ at the midpoint $M$. Since $O M=4-2=2$, we find from the Pythagorean theorem that

$$
A M^{2}=4^{2}-2^{2}=12 \Longrightarrow A M=2 \sqrt{3}
$$

and that $\angle A O M=60^{\circ}$. The area $S$ of the blue-shaded region is

$$
\begin{aligned}
S & =(\text { the area of the circle sector } \triangle O A B)-(\text { the area of the triangle } \triangle O A B) \\
& =\frac{1}{3} \cdot \pi \cdot 4^{2}-\frac{1}{2} \cdot 4 \sqrt{3} \cdot 2 \\
& =\frac{16 \pi}{3}-4 \sqrt{3} .
\end{aligned}
$$

and the volume $V$ of water is

$$
\begin{aligned}
V & =(\text { the area of cross-section }) \times(\text { height }) \\
& =\left(\frac{16 \pi}{3}-4 \sqrt{3}\right) \cdot 9 \\
& =48 \pi-36 \sqrt{3}
\end{aligned}
$$

26. Detective Leblanc interviews three suspects, Agatha, Conan and Ronald. Each one of them is either a murder or an innocent suspect. The murderers all lied and the innocent
suspects all told the truth. The following are the statements made by the suspects:

- Agatha: "Conan is the only murderer."
- Conan: "Agatha and Ronald are both innocent."
- Ronald: "Agatha and Conan are both murderers."

Which of the following is true?
a) Conan is the only murderer
b) Ronald is the only murderer
c) Agatha is the only innocent suspect.
d) Ronald is the only innocent suspect.
e) All three are murderers.

## Answer: d)

Solution) One method is to simply go through all five answers and verify that in all cases but d) either an innocent suspect lies or a murderer tells the truth, which is not allowed by the rules.

Alternatively we can reason like following: Suppose Agatha were innocent. Then Conan would be the only murderer, but this would imply that Ronald is innocent. That would mean that Conan is telling the truth, which as a murderer he would not do. So we conclude that Agatha is a murderer, and we must determine if there were multiple murderers. Since Conan says that Agatha is innocent, he is lying and must also be a murderer. But then Ronald is telling the truth, so he is innocent (and in particular, he is the only innocent suspect.)
27. Two tangents to a circle intersect each other at a point $A$ outside of the circle. The tangent lines intersect the circle at points $B$ and $C$.


The points $B$ and $C$ divide the circle up into two arcs, which have a ratio of lengths of $3: 5$. What is the measure of the angle $\angle B A C$ ?
a) $30^{\circ}$
b) $45^{\circ}$
c) $60^{\circ}$
d) $75^{\circ}$
e) $90^{\circ}$

## Answer: b)

Solution) Let $O$ be the center of the circle, $\theta$ the size of the larger arc and $\phi$ the size of the smaller arc.


We begin by finding the measures of the arcs in degrees. We then have

$$
\frac{\theta}{\phi}=\frac{3}{5} \quad \text { and } \quad \theta+\phi=360^{\circ} \Longrightarrow \phi=225^{\circ} \quad \text { and } \quad \theta=135^{\circ}
$$

Now the measure $\theta=135^{\circ}$ of the smaller arc is the same as the measure of the central angle $B O C$. By the tangent theorem, both $A B O$ and $A C O$ are right angles. Since the angle sum of the convex quadrilateral $A B O C$ must be $360^{\circ}$, we find that

$$
B A C=360^{\circ}-90^{\circ}-90^{\circ}-135^{\circ}=45^{\circ} .
$$

28. If $x, y$ are two distinct positive integers satisfying

$$
\frac{1}{x}+\frac{1}{y}=\frac{1}{37}
$$

then what is $x+y$ ?
a) 345
b) 756
c) 867
d) 1444
e) 2023

## Answer: d)

Solution) Since the equation is symmetric in $x, y$, we may assume that $x>y$. Observe that
$\frac{1}{x}+\frac{1}{y}=\frac{1}{37} \Longrightarrow 37(x+y)=x y \Longrightarrow 37^{2}-37(x+y)+x y=37^{2} \Longrightarrow(x-37)(y-37)=37^{2}$
Using the fundamental theorem of arithmetic, we find that

$$
(x-37)(y-37)=37^{2} \Longrightarrow\left\{\begin{array}{l}
x-37=37^{2} \text { and } y-37=1 \text { or } \\
x-37=-1 \text { and } y-37=-37^{2}
\end{array}\right.
$$

The first pair gives $(x, y)=(1406,38)$ and $x+y=1444$, and the second pair does not produce positive integer solution for $x, y$.
29. The largest real solution $x$ to the equation

$$
\log _{2} x+\log _{4} x+\log _{8} x=\left(\log _{2} x\right)\left(\log _{4} x\right)\left(\log _{8} x\right)
$$

lies in which of the following intervals?
a) $[1,2)$
b) $[2,4)$
c) $[4,8)$
d) $[8,16)$
e) $[16,32)$

## Answer: d)

Solution) Using the change of base formula

$$
\log _{a} b=\frac{\log _{b} b}{\log _{c} a} \quad \text { for all } c>0
$$

we rewrite the given equation so that all terms have base 2 :

$$
\frac{\log _{2} x}{\log _{2} 2}+\frac{\log _{2} x}{\log _{2} 4}+\frac{\log _{2} x}{\log _{2} 8}=\frac{\log _{2} x}{\log _{2} 2} \cdot \frac{\log _{2} x}{\log _{2} 4} \cdot \frac{\log _{2} x}{\log _{2} 8}
$$

If $x \neq 1$, we divide the both sides by $\log _{2} x$ to obtain

$$
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{\left(\log _{2} x\right)^{2}}{1 \cdot 2 \cdot 3} \Longrightarrow\left(\log _{2} x\right)^{2}=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right) \cdot 6=11 \Longrightarrow \log _{2} x= \pm \sqrt{11}
$$

The largest real solution is $x=2^{\sqrt{11}}$ which is between $2^{3}$ and $2^{4}$ because

$$
3=\sqrt{9}<\sqrt{11} \leq \sqrt{16}=4
$$

The largest $x=2^{\sqrt{11}}$ is in interval $[8,16)$.
30. Consider a cone of revolution with inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of the bases lies in the base of the cone. Let $V_{1}$ be the volume of the cone and $V_{2}$ the volume of the cylinder.

the axis of rotation
Find the smallest possible number $k$ for which $V_{1}=k V_{2}$.
a) 1
b) $\frac{4}{3}$
c) $\frac{5}{2}$
d) $\frac{7}{8}$
e) $\frac{9}{5}$

## Answer: b)

Solution) Let $B M$ lie on the axis of rotation as in the figure, and let $A, B$ be the vertices of the triangle in the side view. Also denote by $O$ the center of the sphere.

the axis of rotation

the side view

We introduce the following variables:

- $r=$ the radius of the base of the cone.
- $s=$ the radisu of the inscribed sphere.
- $\theta=$ the angle $\angle O A M$

Since $A M$ and $A B$ are two tangents to the sphere from $A$, we have $\angle O A B=\angle O A M=\theta$. The volume of the cone is

$$
h=r \tan 2 \theta \Longrightarrow V_{1}=\frac{1}{3} \pi r^{2} h=\frac{1}{3} \pi r^{3} \tan 2 \theta
$$

and the volume of the cylinder is

$$
s=r \tan \theta \Longrightarrow V_{2}=\pi s^{2}(2 s)=2 \pi r^{3} \tan ^{3} \theta
$$

Using $\tan 2 \theta=\frac{2 \tan \theta}{1-\tan ^{2} \theta}$, we can write

$$
k=\frac{V_{1}}{V_{2}}=\frac{\tan 2 \theta}{6 \tan ^{3} \theta}=\frac{1}{3 \tan ^{2} \theta\left(1-\tan ^{2} \theta\right)} \Longrightarrow \tan ^{4} \theta-\tan ^{2} \theta+\frac{1}{3 k}=0
$$

Set $T=\tan ^{2} \theta$. For the equation $T^{2}-T+\frac{1}{3 k}=0$ to have (positive) real roots, we must have the discriminant $D=1-\frac{4}{3 k} \geq 0$, or equivalently, $k \geq \frac{4}{3}$. The smallest possible value $k=\frac{4}{3}$ actually occurs when

$$
T=\tan ^{2} \theta=\frac{1}{2} \Longrightarrow \tan \theta=\frac{1}{\sqrt{2}}
$$

or equivalently when the ratio is $s: r=1: \sqrt{2}$.
Tie breaker) Eight identical sheets of paper have been placed overlapping.

| B | F |  |
| :---: | :---: | :---: |
| $C$ |  |  |
|  |  |  |
|  |  |  |

The last sheet to be placed was H because it is shown completely. Fill up the blanks for the the eight pieces in the order placed, ending with H .

(Fill up as many blanks as you can to break any possible ties.)
Solution) We work "backwards". Since the 7-th sheet must be partially covered by H, only F and D are possible candidates for th 7th sheet.

| B | F | H |
| :---: | :---: | :---: |
| C |  |  |
| G | E | D |
|  |  | A |

The 7-th sheet cannot be F because F is covered by another letter C and so 7-th letter must be D.

| B | F | H |
| :---: | :---: | :---: |
| C |  |  |
| G | E | D |
|  |  | A |

The 6-th sheet must be partially covered by D and so it must be A . Continuing in this fashion, we find that the answer is

| B | $F$ | C | $G$ | $E$ | $A$ | D | H |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |
| 1st | 2 nd | 3 rd | 4th | 5th | 6th | 7th | 8 th |


[^0]:    ${ }^{1}$ American Mathematics Competitions
    ${ }^{2}$ Mankato Mathematics Competition

